

LOGARITHMICALLY COMPLETELY MONOTONIC RATIOS OF MEAN VALUES AND AN APPLICATION

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ABSTRACT. In the article, some strictly Logarithmically completely monotonic ratios of mean values are presented.

A function f is said to be completely monotonic on an interval I , if f has derivatives of all orders on I and satisfies

$$(-1)^n f^{(n)}(x) \geq 0 \quad (1)$$

for $x \in I$ and $n \geq 0$. If inequality (1) is strict, then f is said to be strictly completely monotonic on I .

Completely monotonic functions have remarkable applications in different mathematical branches. For instance, they play a role in potential theory [3], probability theory [4, 7, 10], physics [5], numerical and asymptotic analysis [8, 18], and combinatorics [1]. A detailed collection of the most important properties of completely monotonic functions can be found in [17, Chapter IV], and in an abstract in [2].

A positive function f is said to be logarithmically completely monotonic on an interval I if its logarithm $\ln f$ satisfies

$$(-1)^n [\ln f(x)]^{(n)} \geq 0 \quad (2)$$

for $x \in I$ and $n \in \mathbb{N}$. If inequality (2) is strict, then f is said to be strictly logarithmically completely monotonic.

The terminology “(strictly) logarithmically completely monotonic function” was named first by F. Qi, B.-N. Guo and Ch.-P. Chen in [11, 12, 13]. It was also showed in these papers that a (strictly) logarithmically completely monotonic function is also (strictly) completely monotonic.

The generalized logarithmic mean or Stolarsky mean $L_r(a, b)$ of two positive numbers a and b was introduced in [9, 15, 16] and [6, p. 6] for $a = b$ by $L_r(a, b) = a$ and for $a \neq b$ by

$$L_r(a, b) = \left(\frac{b^{r+1} - a^{r+1}}{(r+1)(b-a)} \right)^{1/r}, \quad r \neq -1, 0; \quad (3)$$

$$L_{-1}(a, b) = \frac{b-a}{\ln b - \ln a}; \quad (4)$$

$$L_0(a, b) = \frac{1}{e} \left(\frac{b^b}{a^a} \right)^{1/(b-a)}. \quad (5)$$

2000 *Mathematics Subject Classification.* Primary 26A48, Secondary 30E20.

Key words and phrases. Completely monotonic function, mean, ratio, integral representation.

The authors were supported in part by NSF (#10001016) of China, SF for the Prominent Youth of Henan Province (#0112000200), SF of Henan Innovation Talents at Universities, Doctor Fund of Henan Polytechnic University, China.

Here $L_{-1}(a, b) \triangleq L(a, b)$ and $L_0(a, b) \triangleq I(a, b)$ are the logarithmic and identric means, respectively. When $a \neq b$, $L_r(a, b)$ is a strictly increasing function of r . Further,

$$L_1(a, b) \triangleq A(a, b), \quad L_{-2}(a, b) \triangleq G(a, b), \quad (6)$$

where A and G are the arithmetic and geometric means, respectively.

For $a \neq b$, the following well known inequalities hold

$$H(a, b) < G(a, b) < L(a, b) < I(a, b) < A(a, b), \quad (7)$$

where H is the harmonic mean.

In this paper, the (logarithmically) complete monotonicity of some ratios of mean values are obtained. Our main results are as follows.

Theorem 1. *The ratios*

$$\frac{A(x, x+1)}{I(x, x+1)}, \quad \frac{A(x, x+1)}{G(x, x+1)} = \frac{G(x, x+1)}{H(x, x+1)}, \quad (8)$$

$$\frac{A(x, x+1)}{H(x, x+1)}, \quad \frac{I(x, x+1)}{G(x, x+1)}, \quad \frac{I(x, x+1)}{H(x, x+1)} \quad (9)$$

of mean values A , G , H and I are strictly logarithmically completely monotonic in $(0, \infty)$ and the ratio

$$\frac{A(x, x+1)}{L(x, x+1)} \quad (10)$$

is strictly completely monotonic in $(0, \infty)$.

Proof. Define for $x > 0$

$$\phi_{A/I}(x) = \ln \frac{A(x, x+1)}{I(x, x+1)} = \ln \frac{x+1/2}{(x+1)(1+1/x)^x}. \quad (11)$$

Differentiating directly, using the following representations for $x > 0$, $s \geq 0$ and $n \in \mathbb{N}$

$$\ln x = \int_0^\infty \frac{e^{-t} - e^{-xt}}{t} dt, \quad (12)$$

$$\frac{1}{(x+s)^n} = \frac{1}{(n-1)!} \int_0^\infty t^{n-1} e^{-(x+s)t} dt \quad (13)$$

and the power series expansion of $te^{t/2} - e^t + 1$ at 0, we conclude that

$$\begin{aligned} (-1)^n \phi_{A/I}^{(n)}(x) &= - \int_0^\infty (te^{t/2} - e^t + 1) t^{n-2} e^{-(x+1)t} dt \\ &= \sum_{k=3}^\infty \left(\frac{1}{k} - \frac{1}{2^{k-1}} \right) \frac{1}{(k-1)!} \int_0^\infty t^{n+k-2} e^{-(x+1)t} dt \\ &> 0. \end{aligned} \quad (14)$$

This means that the ratio $\frac{A(x, x+1)}{I(x, x+1)}$ is strictly logarithmically monotonic in $(0, \infty)$.

Define for $x > 0$

$$\phi_{A/G}(x) = \ln \frac{A(x, x+1)}{G(x, x+1)} = \ln \left(x + \frac{1}{2} \right) - \frac{1}{2} \ln x - \frac{1}{2} \ln(x+1), \quad (15)$$

then, by argument as above, we have for any nonnegative integer n

$$\begin{aligned} (-1)^n \phi_{A/G}^{(n)}(x) &= \frac{1}{2} \int_0^\infty (e^t + 1 - 2e^{t/2}) t^{n-1} e^{-(x+1)t} dt \\ &= \frac{1}{2} \sum_{k=2}^\infty \left(1 - \frac{1}{2^{k-1}}\right) \frac{1}{k!} \int_0^\infty t^{n+k-1} e^{-(x+1)t} dt \\ &> 0. \end{aligned} \quad (16)$$

This reveals that the ratio $\frac{A(x, x+1)}{G(x, x+1)}$ is strictly logarithmically completely monotonic in $(0, \infty)$.

Define for $x > 0$

$$\phi_{A/H}(x) = \ln \frac{A(x, x+1)}{H(x, x+1)} = 2 \ln \left(x + \frac{1}{2}\right) - \ln x - \ln(x+1), \quad (17)$$

then we have for nonnegative integer n

$$\begin{aligned} (-1)^n \phi_{A/H}^{(n)}(x) &= \int_0^\infty (e^t - 2e^{t/2} + 1) t^{n-1} e^{-(x+1)t} dt \\ &= \sum_{k=2}^\infty \left(1 - \frac{1}{2^{k-1}}\right) \frac{1}{k!} \int_0^\infty t^{n+k-1} e^{-(x+1)t} dt \\ &> 0. \end{aligned} \quad (18)$$

Therefore, it follows that the ratio $\frac{A(x, x+1)}{H(x, x+1)}$ is strictly logarithmically completely monotonic in $(0, \infty)$.

Define for $x > 0$

$$\phi_{I/G}(x) = \ln \frac{I(x, x+1)}{G(x, x+1)} = x \ln \left(1 + \frac{1}{x}\right) + \frac{1}{2} \ln(x+1) - \frac{1}{2} \ln x - 1. \quad (19)$$

By standard argument above, differentiation for nonnegative integer n yields

$$\begin{aligned} (-1)^n \phi_{I/G}^{(n)}(x) &= \frac{1}{2} \int_0^\infty (2e^t - te^t - t - 2) t^{n-2} e^{-(x+1)t} dt \\ &= \frac{1}{2} \sum_{k=3}^\infty \frac{k-2}{k!} \int_0^\infty t^{n+k-2} e^{-(x+1)t} dt \\ &> 0. \end{aligned} \quad (20)$$

This shows that the ratio $\frac{I(x, x+1)}{G(x, x+1)}$ is also strictly logarithmically completely monotonic in $(0, \infty)$.

Define for $x > 0$

$$\phi_{I/H}(x) = \ln \frac{I(x, x+1)}{H(x, x+1)} = x \ln \left(1 + \frac{1}{x}\right) + \ln \left(x + \frac{1}{2}\right) - \ln x - 1. \quad (21)$$

By the same procedure as above, we obtain for $n \in \mathbb{N}$

$$\begin{aligned} (-1)^n \phi_{I/H}^{(n)}(x) &= \int_0^\infty (e^t + te^{t/2} - te^t - t - 1) t^{n-2} e^{-(x+1)t} dt \\ &= \sum_{k=3}^\infty \left(1 - \frac{1}{k} - \frac{1}{2^{k-1}}\right) \frac{1}{(k-1)!} \int_0^\infty t^{n+k-2} e^{-(x+1)t} dt \\ &> 0. \end{aligned} \quad (22)$$

Thus, it is proved that $\frac{I(x,x+1)}{H(x,x+1)}$ is also strictly logarithmically completely monotonic in $(0, \infty)$.

Define for $x > 0$

$$\phi_{A/L}(x) = \frac{A(x, x+1)}{L(x, x+1)} = \left(x + \frac{1}{2}\right) \ln \left(1 + \frac{1}{x}\right). \quad (23)$$

Straightforward differentiating, using formulas (12) and (13) and expanding the function $2e^t - te^t - t - 2$ at 0 yields

$$\begin{aligned} (-1)^n \phi_{A/L}^{(n)}(x) &= -\frac{1}{2} \int_0^\infty (2e^t - te^t - t - 2)t^{n-2} e^{-(x+1)t} dt \\ &= \frac{1}{2} \sum_{k=3}^\infty \frac{k-2}{k!} \int_0^\infty t^{n+k-2} e^{-(x+1)t} dt \\ &> 0. \end{aligned} \quad (24)$$

This tell us that the ratio $\frac{A(x,x+1)}{L(x,x+1)}$ is strictly completely monotonic in $(0, \infty)$.

The proof is complete. \square

In the final, as an applications of Theorem 1, we give the following remark.

Remark 1. As stated above, a strictly logarithmically completely monotonic function is also strictly completely monotonic. As a result, we deduce from Theorem 1 that

$$e \left(1 - \frac{1}{2x+1}\right) < e \sqrt{\frac{x}{x+1}} < \left(1 + \frac{1}{x}\right)^x < e \left(1 - \frac{1}{2x+2}\right) \quad (25)$$

for $x > 0$. Inequality (25) can be found in [14, 19].

By using the right-hand side of (25), Yang in [19] obtained a strengthened Hardy's inequality:

$$\sum_{n=1}^\infty \lambda_{n+1} (a_1^{\lambda_1} a_2^{\lambda_2} \cdots a_n^{\lambda_n})^{1/\Lambda_n} < e \sum_{n=1}^\infty \left[1 - \frac{\lambda_n}{2(\Lambda_n + \lambda_n)}\right] \lambda_n a_n, \quad (26)$$

where $0 < \lambda_{n+1} \leq \lambda_n$, $\Lambda_n = \sum_{m=1}^n \lambda_m$ and $a_n \geq 0$ for $n \in \mathbb{N}$, $0 < \sum_{n=1}^\infty \lambda_n a_n < \infty$.

In particular, if setting $\lambda_n \equiv 1$, then (26) becomes the following strengthened Carleman's inequality [19]:

$$\sum_{n=1}^\infty (a_1 a_2 \cdots a_n)^{1/n} < e \sum_{n=1}^\infty \left[1 - \frac{1}{2(n+1)}\right] a_n. \quad (27)$$

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