Survey on the numerical methods for ODE's using the sequence of successive approximations

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Abstract

Here we present the error estimations in the numerical methods for Cauchy problems corresponding to ODE's, which use the sequence of successive approximations and quadrature rules such as trapezoidal rule and perturbed trapezoidal rule.

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1 Introduction

There are many results concerning to the numerical methods for ODE's. The properties of the sequence of successive approximations and his application to the numerical methods for ODE's was obtained in [1], [5], [17], [20], [21], [22] and [6].Some of the results in the numerical methods for ODE's obtained in the last 30 years can be found in [4], [9]-[16], [18], [23], [24], [26]-[30]. Here, we continue the work from [6] and [7].

Let $a, b > 0$, $x_0, y_0 \in \mathbb{R}$ and $f : [x_0 - a, x_0 + a] \times [y_0 - b, y_0 + b] \to \mathbb{R}$ a continuous function which verify the Lipschitz condition in the second argument. We consider the initial value problem :

$$\begin{cases} y' = f(x, y) \\ y(x_0) = y_0 \end{cases} \tag{1}$$

having in the space $C^1[x_0 - h, x_0 + h]$ an unique solution, which can be obtained by the successive approximations method starting from any $y \in C^1[x_0 - h, x_0 + h]$ (in particular, we can start from y_0), where $h = \min(a, \frac{b}{M})$ and $M \in \mathbb{R}_+$ is such that $|f(x, y)| \leq M$, $\forall (x, y) \in [x_0 - a, x_0 + a] \times [y_0 - b, y_0 + b]$.

The Lipschitz condition in the second argument is: $\exists L > 0$ such that

$$|f(x, u) - f(x, v)| \leq L\,|u - v|\,,\ \forall x \in [x_0 - a, x_0 + a], \forall u, v \in [y_0 - b, y_0 + b]. \tag{2}$$

The Cauchy's problem (1) is equivalent with the following Volterra integral equation,

$$y(x) = y_0 + \int\limits_{x_0}^{x} f(s, y(s))ds. \tag{3}$$

Applying the Banach's fixed point principle to the integral equation (3) we obtain the sequence of successive approximations,

$$y_0(x) = y_0, \quad \forall x \in [x_0, x_0 + h] \tag{4}$$

$$y_m(x) = y_0 + \int\limits_{x_0}^{x} f(s, y_{m-1}(s))ds, \ \forall x \in [x_0, x_0 + h], \forall m \in \mathbb{N}^*.$$

In [20], at pages 84-95, D.V.Ionescu prove the uniform convergence on $I = [x_0 - h, x_0 + h]$, of the sequence $(y_m)_{m\in\mathbb{N}}$ given in (4). Using the sequence (4) and a uniform partition of $[x_0, x_0 + h]$, D.V.Ionescu give a numerical method to approximate the solution of (1). In this method he apply the classical trapezoidal quadrature rule (for functions with continuous second order derivatives) in [22], to compute the integrals from (4) on the knots of the uniform partition.

In [14] is used the sequence of successive approximations and the classical quadrature trapezoidal rule for functions with continuous second order derivatives to approximate the solutions of Fredholm and Volterra integral equations.

In [21], using the quadrature formula of K.Petr (see [25]),

$$\int\limits_{a}^{b} f(x)dx = \frac{b-a}{2} \cdot [f(a) + f(b)] - \frac{(b-a)^2}{12} \cdot [f'(b) - f'(a)] + R(f) \tag{5}$$

with

$$|R(f)| \leq \frac{(b-a)^5}{720} \cdot \left\| f^{IV} \right\| \tag{6}$$

D.V.Ionescu obtains a numerical method for the problem (1) in which the remainder estimation is obtained by the method from [20] and [22] and by the inequality (6). The inequality (6) holds for functions $f \in C^4(D)$, where $D = [x_0 - a, x_0 + a] \times [y_0 - b, y_0 + b]$. Unfortunately, the fourth derivative $\frac{d^4 f(t,x(t))}{dt^4}$ have 13 terms, and therefore the remainder containing more terms, increase. This is the reason for which we consider that we can stop to the third derivative in the estimation of the remainder, trying to use an adequate quadrature formula.

In [6] and [7], we have used a recent remainder estimation in the formula (5) obtained by N.S.Barnett and S.S.Dragomir in [3] and .[2] We sumarize the results from [2] and [3] in the following inequality,

$$\left| \int\limits_{a}^{b} f(x)dx - \frac{b-a}{2}[f(a) + f(b)] + \frac{(b-a)^2}{12}[f'(b) - f'(a)] \right| \leq$$

$$\leq \begin{cases} \frac{(b-a)^4}{160} \cdot \|f'''\|, \text{ if } & f \in C^3[a,b] \\ \frac{(b-a)^5}{720} L, \quad \text{if} & f''' \in Lip[a,b] \\ \frac{(b-a)^5}{720} \cdot \left\|f^{IV}\right\|, \text{ if } & f \in C^4[a,b], \end{cases} \tag{7}$$

where L is the Lipschitz constant of the third derivative f''' and

$$Lip[a,b] = \{g \in C[a,b] : g \text{ is Lpischitzian}\}.$$

Using a uniform partition of the interval $[a,b]$,

$$\Delta : a = x_0 < x_1 < ... < x_{n-1} < x_n = b, \quad x_i = a + \frac{i(b-a)}{n}, \forall i = \overline{0,n},$$

in [2], N.S.Barnett and S.S.Dragomir obtain the following quadrature rule,

$$\int_a^b f(x)dx = \frac{b-a}{2n}[f(a)+2\sum_{i=1}^{n-1} f(x_i)+f(b)]-\frac{(b-a)^2}{12n^2}[f'(b)-f'(a)]+R_n(f) \tag{8}$$

with the remainder estimation,

$$|R_n(f)| \leq \frac{(b-a)^4}{160n^3} \cdot \|f'''\|. \tag{9}$$

If f''' is Lipschitzian with the constant L, then the remainder estimation, according to [3], is,

$$|R_n(f)| \leq \frac{(b-a)^5 L}{720n^4}, \tag{10}$$

and if $f \in C^4[a,b]$, then, according to [3] and [21], the following inequality, which is used in [21], holds:

$$|R_n(f)| \leq \frac{(b-a)^5}{720n^4} \cdot \left\|f^{IV}\right\|. \tag{11}$$

In [6] and [7] we have obtained the error estimation in the numerical method for ODE's which use the sequence of successive approximations and the quadrature rule (8) with the remainder estimation (9) and (10), respectively.

On the other hand, here we use also the trapezoidal quadrature rule obtained in [13] and [8]:

$$\int_a^b f(x)dx = \frac{b-a}{2n} \cdot [f(a) + 2\sum_{i=1}^{n-1} f(a + \frac{i(b-a)}{n}) + f(b)] + R_n(f), \tag{12}$$

with the remainder estimation,

$$|R_n(f)| \leq \begin{cases} \frac{(b-a)^2 L}{4n}, & \text{if} \quad f \in Lip[a,b] \quad (\text{ in [13] }) \\ \frac{(b-a)^2 \cdot \|f'\|}{4n}, & \text{if} \quad f \in C^1[a,b] \quad (\text{ in [13] }) \\ \frac{(b-a)^3 L'}{12n^2}, & \text{if} \quad f' \in Lip[a,b] \quad (\text{ in [8] }) \\ \frac{(b-a)^3 \|f''\|}{12n^2}, & \text{if} \quad f \in C^2[a,b], \end{cases} \tag{13}$$

where L is the Lipschitz constant of f and L' is the Lipschitz constant of f'.

2 Numerical method using the trapezoidal quadrature rule

Let Δ an uniform partition of $[x_0, x_0 + h]$,

$$\Delta : x_0 < x_1 < ... < x_{n-1} < x_n = x_0 + h, \quad x_i = x_0 + \frac{ih}{n}, \forall i = \overline{1, n}.$$

On the knots $x_i, i = \overline{1, n}$, the sequence of successive approximations (4) is,

$$\begin{gathered} y_0(x_i) = y_0, \quad \forall i = \overline{0, n}. \\ y_m(x_i) = y_0 + \int_{x_0}^{x_i} f(s, y_{m-1}(s))ds, \ \forall i = \overline{0, n}, \forall m \in \mathbb{N}^*. \end{gathered} \tag{14}$$

We define the functions,

$$F_m : [x_0, x_0 + h] \to \mathbb{R}, \ F_m(x) = f(x, y_m(x)), \ \forall x \in [x_0, x_0 + h], \forall m \in \mathbb{N}.$$

To compute the integrals from (14) we use the quadrature rule (12)-(13), and obtain:

$$\begin{gathered} y_m(x_i) = y_0 + \int_{x_0}^{x_i} F_{m-1}(s)ds = y_0 + \frac{h}{2n}[F_{m-1}(x_0) + 2\sum_{j=1}^{i-1} F_{m-1}(x_j) + \\ + F_{m-1}(x_i)] + R_{m,i}(f) = y_0 + \frac{h}{2n}[f(x_0, y_0) + 2\sum_{j=1}^{i-1} f(x_j, y_{m-1}(x_j)) + \\ + f(x_i, y_{m-1}(x_i))] + R_{m,i}(f), \quad \forall i = \overline{1, n}, \forall m \in \mathbb{N}^*, \end{gathered} \tag{15}$$

and the remainder estimation is,

$$|R_{m,i}(f)| \leq \begin{cases} \frac{h^2}{4n} \cdot L, \quad \forall i = \overline{1,n}, \forall m \in \mathbb{N}^*, \\ \\ \frac{h^2}{4n} \cdot \left\| F'_{m-1} \right\|, \quad \forall i = \overline{1,n}, \forall m \in \mathbb{N}^* \\ \\ \frac{h^3}{12n^2} \cdot L', \quad \forall i = \overline{1,n}, \forall m \in \mathbb{N}^*, \\ \\ \frac{h^3}{12n^2} \cdot \left\| F''_{m-1} \right\|, \quad \forall i = \overline{1,n}, \forall m \in \mathbb{N}^* \end{cases} \tag{16}$$

where L' is an upper bound of the Lipschitz constants of the functions F'_{m-1}, $m \in \mathbb{N}^*$ (and L is an upper bound of the Lipschitz constants of the functions F_{m-1}, $m \in \mathbb{N}^*$).

The relations (15) and (16) lead to the following algorithm :

$$y_m(x_0) = y_0,\ \forall m \in \mathbb{N} \quad \text{and} \quad y_0(x_i) = y_0,\ \forall i = \overline{0,n},$$

$$y_1(x_i) = y_0 + \frac{h}{2n}[f(x_0, y_0) + 2\sum_{j=1}^{i-1} f(x_j, y_0(x_j)) + f(x_i, y_0(x_i))] +$$

$$+R_{1,i}(f) = y_0 + \frac{h}{2n}[f(x_0, y_0) + 2\sum_{j=1}^{i-1} f(x_j, y_0) + f(x_i, y_0)] +$$

$$+R_{1,i}(f) = \overline{y_1(x_i)} + R_{1,i}(f),\ \forall i = \overline{1,n} \tag{17}$$

$$y_2(x_i) = y_0 + \frac{h}{2n}[f(x_0, y_0) + 2\sum_{j=1}^{i-1} f(x_j, \overline{y_1(x_j)} + R_{1,j}(f)) + f(x_i, \overline{y_1(x_i)} +$$

$$+R_{1,i}(f))] + R_{2,i}(f) = y_0 + \frac{h}{2n}[f(x_0, y_0) + 2\sum_{j=1}^{i-1} f(x_j, \overline{y_1(x_j)}) +$$

$$+f(x_i, \overline{y_1(x_i)})] + \overline{R_{2,i}(f)} = \overline{y_2(x_i)} + \overline{R_{2,i}(f)},\ \forall i = \overline{1,n}. \tag{18}$$

By induction, for $m \in \mathbb{N}$, $m \geq 3$, we obtain :

$$y_m(x_i) = y_0 + \frac{h}{2n}[f(x_0, y_0) + 2\sum_{j=1}^{i-1} f(x_j, \overline{y_{m-1}(x_j)} + \overline{R_{m-1,j}(f)}) +$$

$$+f(x_i, \overline{y_{m-1}(x_i)} + \overline{R_{m-1,j}(f)})] + R_{m,i}(f) =$$

$$= y_0 + \frac{h}{2n}[f(x_0, y_0) + 2\sum_{j=1}^{i-1} f(x_j, \overline{y_{m-1}(x_j)}) + f(x_i, \overline{y_{m-1}(x_i)})] +$$

$$+\overline{R_{m,i}(f)} = \overline{y_m(x_i)} + \overline{R_{m,i}(f)},\ \forall i = \overline{1,n}, \qquad \forall .m \in \mathbb{N}, m \geq 3. \tag{19}$$

Let $\gamma \geq 0$ such that,

$$|f(t,u) - f(x,u)| \leq \gamma |t-x|, \ \forall t, x \in [x_0 - a, x_0 + a], \forall u \in [y_0 - b, y_0 + b]. \quad (20)$$

If $f \in C^1([x_0 - a, x_0 + a] \times [y_0 - b, y_0 + b])$, since

$$F'_m(x) = \frac{\partial f}{\partial x}(x, y_m(x)) + \frac{\partial f}{\partial y}(x, y_m(x)) \cdot y'_m(x) =$$

$$= \frac{\partial f}{\partial x}(x, y_m(x)) + \frac{\partial f}{\partial y}(x, y_m(x)) \cdot f(x, y_{m-1}(x)),$$

follows that

$$\left\| F'_{m-1} \right\| \leq M_1(M+1), \quad \forall m \in \mathbb{N}^*.$$

where,

$$M_1 = \max(\left\| \frac{\partial f}{\partial x} \right\|, \left\| \frac{\partial f}{\partial y} \right\|).$$

Theorem 1 *If $f \in C([x_0 - a, x_0 + a] \times [y_0 - b, y_0 + b])$ and $hL < 1$, in the conditions (2) and (20) then the functions F_m, $m \in \mathbb{N}$, are Lipschitzian with the Lipschitz constant $\gamma + LM$, and the sequence $(\overline{y_m(x_i)})_{m \in \mathbb{N}^*}$ given by (19) approximate the solution of the Cauchy problem (1) on the knots t_i, $i = \overline{1,n}$, with the error estimation :*

$$\left| y^*(x_i) - \overline{y_m(x_i)} \right| \leq \frac{(Lh)^m}{1 - Lh}(|y_0| + Mh) + (\frac{1}{1 - hL}) \cdot \frac{h^2}{4n}(\gamma + LM), \forall i = \overline{1,n}, m \in \mathbb{N}^*. \quad (21)$$

Proof. We have successively, by induction,

$$|F_0(t_1) - F_0(t_2)| \leq \gamma |t_1 - t_2|, \qquad \forall t_1, t_2 \in [x_0, x_0 + h],$$

$$|F_m(t_1) - F_m(t_2)| \leq \gamma |t_1 - t_2| + L |y_m(t_1) - y_m(t_2)| \leq \gamma |t_1 - t_2| +$$

$$+ L \int_{t_1}^{t_2} |f(s, y_{m-1}(s))| \, ds \leq (\gamma + LM) |t_1 - t_2|, \qquad \forall t_1, t_2 \in [x_0, x_0 + h], \ \forall m \in \mathbb{N}^*.$$

From the Banach's fixed point principle we have the following estimation

$$|y^*(x) - y_m(x)| \leq \frac{(Lh)^m}{1 - Lh} \cdot \|y_0 - y_1\|_C, \quad \forall x \in [x_0, x_0 + h], \forall m \in \mathbb{N}^*, \quad (22)$$

in the approximation of the solution of (1) by the sequence of successive approximations, where

$$\|u\|_C = \max\{|u(t)| : t \in [a, b]\}, \quad \forall u \in C[a, b].$$

We observe that $\|y_0 - y_1\|_C \leq |y_0| + Mh$.
From the relations (19) we have

$$\left| y_m(x_i) - \overline{y_m(x_i)} \right| = \left| \overline{R_{m,i}(f)} \right|, \quad \forall i = \overline{1,n}, \quad \forall m \in \mathbb{N}^*.$$

Moreover,

$$\left|\overline{R_{2,i}(f)}\right| \le |R_{2,i}(f)| + \frac{h}{2n}(2L\sum_{j=1}^{i-1}|R_{1,j}(f)| + L\,|R_{1,i}(f)|) \le$$

$$\le |R_{2,i}(f)| + Lh\cdot\frac{h^2}{4n}(\gamma+LM) \le (1+hL)\cdot\frac{h^2}{4n}(\gamma+LM),\ \forall i=\overline{1,n}. \tag{23}$$

and

$$\left|\overline{R_{m,i}(f)}\right| \le |R_{m,i}(f)| + \frac{h}{2n}[L\left|\overline{R_{m-1,i}(f))}\right| + 2L\sum_{j=1}^{i-1}\left|\overline{R_{m-1,j}(f)}\right|] \le$$

$$\le \frac{h^4M'''}{160n^3} + \frac{hL}{2n}(\left|\overline{R_{m-1,i}(f))}\right| + 2\sum_{j=1}^{i-1}\left|\overline{R_{m-1,j}(f)}\right|) \le$$

$$\le [1+hL+...+(hL)^{m-1}]\cdot\frac{h^2}{4n}(\gamma+LM),\ \forall i=\overline{1,n},\ \forall m\in\mathbb{N}^*. \tag{24}$$

If $hL<1$ then

$$\left|\overline{R_{m,i}(f)}\right| \le (\frac{1}{1-hL})\cdot\frac{h^2}{4n}(\gamma+LM),\ \forall i=\overline{1,n},\ \forall m\in\mathbb{N}^*. \tag{25}$$

Since

$$\left|y^*(x_i)-\overline{y_m(x_i)}\right| \le |y^*(x_i)-y_m(x_i)|+\left|y_m(x_i)-\overline{y_m(x_i)}\right|,\quad \forall i=\overline{1,n},\ m\in\mathbb{N}^*,$$

from (22) and (25) follows the inequality (21). ■

Theorem 2 *If $f\in C^1([x_0-a,x_0+a]\times[y_0-b,y_0+b])$ and $hL<1$, then the sequence $(\overline{y_m(x_i)})_{m\in\mathbb{N}^*}$ given by (19) approximate the solution of the Cauchy problem (1) on the knots t_i, $i=\overline{1,n}$, with the error estimation :*

$$\left|y^*(x_i)-\overline{y_m(x_i)}\right| \le \frac{(Lh)^m}{1-Lh}(|y_0|+Mh)+(\frac{1}{1-hL})\cdot\frac{h^2}{4n}M_1(M+1),\forall i=\overline{1,n}, m\in\mathbb{N}^*. \tag{26}$$

Proof. Since

$$\left\|F'_{m-1}\right\| \le M_1(M+1),\quad \forall m\in\mathbb{N}^*,$$

by the inequality (16) in an analogous manner as in the proof of Theorem 1, we obtain

$$\left|\overline{R_{m,i}(f)}\right| \le (\frac{1}{1-hL})\cdot\frac{h^2}{4n}M_1(M+1),\ \forall i=\overline{1,n},\ \forall m\in\mathbb{N}^*$$

which lead to the estimation (26). ■

If $f \in C^1([x_0-a, x_0+a] \times [y_0-b, y_0+b])$ we can consider the Lipschitz conditions for the first partial derivatives :

$$\left|\frac{\partial f}{\partial x}(t_1,u)-\frac{\partial f}{\partial x}(t_2,u)\right| \le L_{10}\left|t_1-t_2\right|, \quad \forall t_1,t_2 \in [x_0,x_0+h], \forall u \in [y_0-b,y_0+b] \tag{27}$$

$$\left|\frac{\partial f}{\partial x}(t,u_1)-\frac{\partial f}{\partial x}(t,u_2)\right| \le L_{01}\left|u_1-u_2\right|, \quad \forall t \in [x_0,x_0+h], \forall u_1,u_2 \in [y_0-b,y_0+b] \tag{28}$$

$$\left|\frac{\partial f}{\partial y}(t_1,u)-\frac{\partial f}{\partial y}(t_2,u)\right| \le L_{12}\left|t_1-t_2\right|, \quad \forall t_1,t_2 \in [x_0,x_0+h], \forall u \in [y_0-b,y_0+b] \tag{29}$$

$$\left|\frac{\partial f}{\partial y}(t,u_1)-\frac{\partial f}{\partial y}(t,u_2)\right| \le L_{21}\left|u_1-u_2\right|, \quad \forall t \in [x_0,x_0+h], \forall u_1,u_2 \in [y_0-b,y_0+b] \tag{30}$$

where $L_{10} \ge 0,\ L_{01} \ge 0,\ L_{12} \ge 0,\ L_{21} \ge 0.$

Theorem 3 *If $f \in C^1([x_0-a, x_0+a] \times [y_0-b, y_0+b])$ and $Lh < 1$, in the conditions (27)-(30) then the functions $F'_m,\ m \in \mathbb{N}$, are Lipschitzian with the Lipschitz constant*

$$L' = L_{10} + M\left(L_{01}+L_{12}\right) + M^2 L_{21} + M_1(\gamma + LM),$$

and the sequence $(\overline{y_m(x_i)})_{m\in\mathbb{N}^*}$ given by (19) approximate the solution of the Cauchy problem (1) on the knots t_i, $i=\overline{1,n}$, with the error estimation :

$$\left|y^*(x_i)-\overline{y_m(x_i)}\right| \le \frac{(Lh)^m}{1-Lh}(|y_0|+Mh)+\left(\frac{1}{1-hL}\right)\cdot\frac{L'h^3}{12n^2}, \quad \forall i=\overline{1,n}, \forall m \in \mathbb{N}^*. \tag{31}$$

Proof. Firstly, we obtain

$$\left|F'_0(t_1)-F'_0(t_2)\right| \le \left(L_{10}+ML_{01}\right)\left|t_1-t_2\right|, \quad \forall t_1,t_2 \in [x_0,x_0+h]$$

and by induction follows for any $t_1,t_2 \in [x_0,x_0+h]$ and $m \in \mathbb{N}^*$ that,

$$\left|F'_m(t_1)-F'_m(t_2)\right| \le \left|\frac{\partial f}{\partial x}(t_1,y_m(t_1))-\frac{\partial f}{\partial x}(t_2,y_m(t_2))\right|+$$

$$+\left|\frac{\partial f}{\partial y}(t_1,y_m(t_1))-\frac{\partial f}{\partial y}(t_2,y_m(t_2))\right|\cdot\left|f(t_1,y_{m-1}(t_1))\right|+\left|\frac{\partial f}{\partial y}(t_2,y_m(t_2))\right|\cdot$$

$$\cdot\left|f(t_1,y_{m-1}(t_1))-f(t_2,y_{m-1}(t_2))\right| \le L_{10}\left|t_1-t_2\right|+L_{01}\left|y_m(t_1)-y_m(t_2)\right|+$$

$$+M(L_{12}\left|t_1-t_2\right|+L_{21}\left|y_m(t_1)-y_m(t_2)\right|)+M_1(\gamma+LM)\left|t_1-t_2\right| \le$$

$$\le \left[L_{10}+ML_{01}+M(L_{12}+ML_{21})+M_1(\gamma+LM)\right]\cdot\left|t_1-t_2\right| =$$

$$= \left[L_{10}+M\left(L_{01}+L_{12}\right)+M^2L_{21}+M_1(\gamma+LM)\right]\cdot\left|t_1-t_2\right| = L'\left|t_1-t_2\right|.$$

From inequality (16), in a similar way as in the Theorem 1, we obtain the estimation (31). ■

If $f \in C^2([x_0 - a, x_0 + a] \times [y_0 - b, y_0 + b])$ we have,

$$F_m''(x) = \frac{\partial^2 f}{\partial x^2}(x, y_m(x)) + 2\frac{\partial^2 f}{\partial x \partial y}(x, y_m(x)) \cdot f(x, y_{m-1}(x)) +$$

$$+ \frac{\partial^2 f}{\partial y^2}(x, y_m(x)) \cdot [f(x, y_{m-1}(x))]^2 + \frac{\partial f}{\partial y}(x, y_m(x)) \cdot [\frac{\partial f}{\partial x}(x, y_{m-1}(x)) +$$

$$+ \frac{\partial f}{\partial y}(x, y_{m-1}(x)) \cdot f(x, y_{m-2}(x))]$$

and therefore

$$|F_m''(x)| \leq M_2(M+1)^2 + M_1^2(M+1) = M'', \quad \forall x \in [x_0, x_0 + h], \tag{32}$$

where

$$M_2 = \max\{\left\|\frac{\partial^2 f}{\partial x^2}\right\|, \left\|\frac{\partial^2 f}{\partial x \partial y}\right\|, \left\|\frac{\partial^2 f}{\partial y^2}\right\|\}.$$

Theorem 4 *If $f \in C^1([x_0 - a, x_0 + a] \times [y_0 - b, y_0 + b])$ and $Lh < 1$ then the sequence $(\overline{y_m(x_i)})_{m \in \mathbb{N}^*}$ given by (19) approximate the solution of the Cauchy problem (1) on the knots t_i, $i = \overline{1, n}$, with the error estimation :*

$$\left|y^*(x_i) - \overline{y_m(x_i)}\right| \leq \frac{(Lh)^m}{1 - Lh}(|y_0| + Mh) + (\frac{1}{1 - hL}) \cdot \frac{M'' h^3}{12n^2}, \quad \forall i = \overline{1, n}, \forall m \in \mathbb{N}^*. \tag{33}$$

Proof. Using the inequality (32) in a similar way as in the Theorem 1, we obtain the estimation (33). ■

3 Numerical methods using the perturbed trapezoidal quadrature rule

Using the perturbed trapezoidal quadrature rule (8), with the remainder estimation (9) and the method of successive approximations we have obtained in [6] the numerical method :

$$y_m(x_i) = y_0 + \frac{h}{2n}[f(x_0, y_0) + 2\sum_{j=1}^{i-1} f(x_j, y_{m-1}(x_j)) + f(x_i, y_{m-1}(x_i))] -$$

$$- \frac{h^2}{12n^2}[\frac{\partial f}{\partial x}(x_i, y_{m-1}(x_i)) + \frac{\partial f}{\partial y}(x_i, y_{m-1}(x_i)) \cdot f(x_i, y_{m-2}(x_i)) -$$

$$- \frac{\partial f}{\partial x}(x_0, y_0) - \frac{\partial f}{\partial y}(x_0, y_0) \cdot f(x_0, y_0)] + R_{m,i}(f), \quad \forall i = \overline{1, n}, \forall m \in \mathbb{N}^*. \tag{34}$$

with the remainder estimation,

$$|R_{m,i}(f)| \leq \frac{h^4}{160n^3} \cdot \left\| F'''_{m-1} \right\|, \quad \forall i = \overline{1,n}, \forall m \in \mathbb{N}^*. \tag{35}$$

After elementary calculus we have,

$$F'''_{m-1}(x) = \frac{\partial^3 f}{\partial x^3}(x, y_m(x)) + 3\frac{\partial^3 f}{\partial x^2 \partial y}(x, y_m(x)) \cdot y'_m(x) + 3\frac{\partial^3 f}{\partial x \partial y^2}(x, y_m(x)) \cdot$$

$$\cdot [y'_m(x)]^2 + \frac{\partial^3 f}{\partial y^3}(x, y_m(x)) \cdot [y'_m(x)]^3 + 3\frac{\partial^2 f}{\partial x \partial y}(x, y_m(x)) \cdot y''_m(x) +$$

$$3\frac{\partial^2 f}{\partial y^2}(x, y_m(x)) \cdot y'_m(x) \cdot y''_m(x) + \frac{\partial f}{\partial y}(x, y_m(x)) \cdot y'''_m(x), \ \forall x \in [x_0, x_0 + h], \ \forall m \in \mathbb{N}.$$

and,

$$y''_m(x) = \frac{\partial f}{\partial x}(x, y_{m-1}(x)) + \frac{\partial f}{\partial y}(x, y_{m-1}(x)) \cdot y'_{m-1}(x),$$

$$y'''_m(x) = \frac{\partial^2 f}{\partial x^2}(x, y_{m-1}(x)) + 2\frac{\partial^2 f}{\partial x \partial y}(x, y_{m-1}(x)) \cdot y'_{m-1}(x) + \frac{\partial^2 f}{\partial y^2}(x, y_{m-1}(x)) \cdot$$

$$\cdot [y'_{m-1}(x)]^2 + \frac{\partial f}{\partial y}(x, y_{m-1}(x)) \cdot y''_{m-1}(x), \ \forall x \in [x_0, x_0 + h], \forall m \in \mathbb{N}^*.$$

$$y'_0(x) = y''_0(x) = y'''_0(x) = 0, \ \forall x \in [x_0, x_0 + h].$$

For

$$\left\| \frac{\partial^\alpha f}{\partial x^{\alpha_1} \partial y^{\alpha_2}} \right\| = \max\{ \left| \frac{\partial^\alpha f(x,y)}{\partial x^{\alpha_1} \partial y^{\alpha_2}} \right| : x \in [x_0, x_0 + h], y \in [y_0 - b, y_0 + b]\}$$

and $|\alpha| \leq 3$, $\alpha_1 + \alpha_2 = |\alpha|$, $\alpha_1, \alpha_2 \in \{0, 1, 2, 3\}$, let

$$M_1 = \max\{ \left\| \frac{\partial f}{\partial x} \right\|, \left\| \frac{\partial f}{\partial y} \right\| \},$$

$$M_2 = \max\{ \left\| \frac{\partial^2 f}{\partial x^2} \right\|, \left\| \frac{\partial^2 f}{\partial x \partial y} \right\|, \left\| \frac{\partial^2 f}{\partial y^2} \right\| \}$$

$$M_3 = \max\{ \left\| \frac{\partial^3 f}{\partial x^3} \right\|, \left\| \frac{\partial^3 f}{\partial x^2 \partial y} \right\|, \left\| \frac{\partial^3 f}{\partial x \partial y^2} \right\|, \left\| \frac{\partial^3 f}{\partial y^3} \right\| \}.$$

So,

$$|F'''_m(x)| \leq M_3 + 3MM_3 + 3M_3M^2 + M_3M^3 + 3M_1M_2(M+1) +$$

$$+3MM_1M_2(M+1) + M_1[M_2 + 2MM_2 + M_2M^2 + M_1^2(M+1) = M_3(M+1)^3 +$$

$$+4M_1M_2(M+1)^2 + M_1^3(M+1) = M''', \forall x \in [x_0, x_0 + h], \forall m \in \mathbb{N},$$

and therefore, the inequality (35) became:

$$|R_{m,i}(f)| \leq \frac{h^4 M'''}{160n^3},\ \forall i = \overline{1,n}, \forall m \in \mathbb{N}^*. \tag{36}$$

The method (34) lead to the algorithm,

$$y_m(x_0) = y_0,\ \forall m \in \mathbb{N} \text{ and } y_0(x_i) = y_0,\ \forall i = \overline{0,n},$$

$$y_1(x_i) = y_0 + \frac{h}{2n}[f(x_0,y_0) + 2\sum_{j=1}^{i-1} f(x_j, y_0(x_j)) + f(x_i, y_0(x_i))] -$$

$$-\frac{h^2}{12n^2}[\frac{\partial f}{\partial x}(x_i, y_0(x_i)) - \frac{\partial f}{\partial x}(x_0,y_0)] + R_{1,i}(f) = y_0 + \frac{h}{2n}[f(x_0,y_0) +$$

$$+2\sum_{j=1}^{i-1} f(x_j, y_0) + f(x_i, y_0)] - \frac{h^2}{12n^2}[\frac{\partial f}{\partial x}(x_i,y_0) - \frac{\partial f}{\partial x}(x_0,y_0)] +$$

$$+R_{1,i}(f) = \overline{y_1(x_i)} + R_{1,i}(f),\ \forall i = \overline{1,n}$$

$$y_2(x_i) = y_0 + \frac{h}{2n}[f(x_0,y_0) + 2\sum_{j=1}^{i-1} f(x_j, \overline{y_1(x_j)} + R_{1,j}(f)) + f(x_i, \overline{y_1(x_i)} + R_{1,i}(f))]$$

$$-\frac{h^2}{12n^2}[\frac{\partial f}{\partial x}(x_i, \overline{y_1(x_i)} + R_{1,i}(f)) + \frac{\partial f}{\partial y}(x_i, \overline{y_1(x_i)} + R_{1,i}(f)) \cdot f(x_i, y_0) - \frac{\partial f}{\partial x}(x_0,y_0)$$

$$-\frac{\partial f}{\partial y}(x_0,y_0) \cdot f(x_0,y_0)] + R_{2,i}(f) = y_0 + \frac{h}{2n}[f(x_0,y_0) + 2\sum_{j=1}^{i-1} f(x_j, \overline{y_1(x_j)}) +$$

$$+f(x_i, \overline{y_1(x_i)})] - \frac{h^2}{12n^2}[\frac{\partial f}{\partial x}(x_i, \overline{y_1(x_i)}) + \frac{\partial f}{\partial y}(x_i, \overline{y_1(x_i)}) \cdot f(x_i, y_0) - \frac{\partial f}{\partial x}(x_0,y_0)$$

$$-\frac{\partial f}{\partial y}(x_0,y_0) \cdot f(x_0,y_0)] + \overline{R_{2,i}(f)} = \overline{y_2(x_i)} + \overline{R_{2,i}(f)},\ \forall i = \overline{1,n}$$

By induction, for $m \in \mathbb{N}$, $m \geq 3$, we have obtained :

$$y_m(x_i) = y_0 + \frac{h}{2n}[f(x_0,y_0) + 2\sum_{j=1}^{i-1} f(x_j, \overline{y_{m-1}(x_j)} + \overline{R_{m-1,j}(f)}) + f(x_i, \overline{y_{m-1}(x_i)} +$$

$$+\overline{R_{m-1,j}(f)})] - \frac{h^2}{12n^2}[\frac{\partial f}{\partial x}(x_i, \overline{y_{m-1}(x_i)} + \overline{R_{m-1,j}(f)}) + \frac{\partial f}{\partial y}(x_i, \overline{y_{m-1}(x_i)} +$$

$$+\overline{R_{m-1,j}(f)}) \cdot f(x_i, \overline{y_{m-2}(x_i)} + \overline{R_{m-2,j}(f)}) - \frac{\partial f}{\partial x}(x_0,y_0) - \frac{\partial f}{\partial y}(x_0,y_0) \cdot f(x_0,y_0)]$$

$$+R_{m,i}(f) = y_0 + \frac{h}{2n}[f(x_0,y_0) + 2\sum_{j=1}^{i-1} f(x_j, \overline{y_{m-1}(x_j)}) + f(x_i, \overline{y_{m-1}(x_i)})] -$$

$$-\frac{h^2}{12n^2}[\frac{\partial f}{\partial x}(x_i,\overline{y_{m-1}(x_i)})+\frac{\partial f}{\partial y}(x_i,\overline{y_{m-1}(x_i)})\cdot f(x_i,\overline{y_{m-2}(x_i)})-\frac{\partial f}{\partial x}(x_0,y_0)-$$

$$-\frac{\partial f}{\partial y}(x_0,y_0)\cdot f(x_0,y_0)]+\overline{R_{m,i}(f)}=\overline{y_m(x_i)}+\overline{R_{m,i}(f)},\ \forall i=\overline{1,n},\ \forall m\geq 3. \quad (37)$$

For the remainder estimation we have obtained (for $m\in\mathbb{N}, m\geq 2$) the inequality:

$$\left|\overline{R_{m,i}(f)}\right|\leq\frac{1-[hL+\frac{h^2}{12n^2}(M_2(M+1)+M_1L)]^m}{1-[hL+\frac{h^2}{12n^2}(M_2(M+1)+M_1L)]}\cdot\frac{h^4M'''}{160n^3}\ ,\ \forall i=\overline{1,n}, \quad (38)$$

which lead to the following result :

Theorem 5 *(in [6]) : If $Lh<\frac{11}{12}$ and $f\in C^3([x_0-a,x_0+a]\times[y_0-b,y_0+b])$ then, for $n\in\mathbb{N}$, $n>h\sqrt[2]{M_2(M+1)+M_1L}$, the sequence $(\overline{y_m(x_i)})_{m\in\mathbb{N}}$ given in (37) approximates on the knots x_i , $i=\overline{1,n}$, the solution y^* of the Cauchy's problem (1) with the error estimation :*

$$\left|y^*(x_i)-\overline{y_m(x_i)}\right|\leq\frac{(Lh)^m}{1-Lh}(|y_0|+Mh)+$$

$$+\frac{h^4M'''}{160n^3(1-[hL+\frac{h^2}{12n^2}(M_2(M+1)+M_1L)])},\ \forall i=\overline{1,n},\forall m\in\mathbb{N},m\geq 2. \quad (39)$$

With the perturbed trapezoidal quadrature rule (8) having the remainder estimation (10), in [7] we have obtained for the numerical method given in (34) and (37) the error estimation :

$$\left|\overline{R_{m,i}(f)}\right|\leq\frac{h^5L'''}{720n^4(1-[hL+\frac{h^2}{12n^2}(M_2(M+1)+M_1L)])},\ \forall i=\overline{1,n},\forall m\geq 2, \quad (40)$$

where $L'''\geq 0$ is the Lipschitz constant of the functions F_m''', $m\in\mathbb{N}$, in the Lipschitz conditions of the third order derivatives

$$\frac{\partial^3 f}{\partial x^3},\quad\frac{\partial^3 f}{\partial x^2\partial y},\quad\frac{\partial^3 f}{\partial x\partial y^2},\quad\frac{\partial^3 f}{\partial y^3}$$

in each argument.

In this case we have obtained in [7] the following result :

Theorem 6 *(in [7]) : If $f\in C^3([x_0-a,x_0+a]\times[y_0-b,y_0+b])$, $Lh<\frac{11}{12}$ and the third order partial derivatives*

$$\frac{\partial^3 f}{\partial x^3},\quad\frac{\partial^3 f}{\partial x^2\partial y},\quad\frac{\partial^3 f}{\partial x\partial y^2},\quad\frac{\partial^3 f}{\partial y^3}$$

are Lipschitzian in each argument then the sequence $(\overline{y_m(x_i)})_{m\in\mathbb{N}}$ *approximates the solution of the initial value problem (1) on the knots* x_i , $i = \overline{1,n}$, *and for* $n \in \mathbb{N}$, $n > h\sqrt[2]{M_2(M+1) + M_1 L}$, *with the error estimation :*

$$\left|y^*(x_i) - \overline{y_m(x_i)}\right| \leq \frac{(Lh)^m}{1-Lh}(|y_0| + Mh) +$$

$$+\frac{h^5 L'''}{720n^4(1-[hL+\frac{h^2}{12n^2}(M_2(M+1)+M_1L)])},\ \forall i = \overline{1,n}, \forall m \in \mathbb{N}, m \geq 2. \tag{41}$$

If $f \in C^4([x_0 - a, x_0 + a] \times [y_0 - b, y_0 + b])$, then for

$$M_4 = \max\{\left\|\frac{\partial^4 f}{\partial x^i \partial y^j}\right\| : i,j = \overline{0,4},\ i+j=4\}$$

and after elementary calculus we obtain the fourth derivative $F_m^{(IV)}(x)$ which has 13 terms,

$$F_m^{(IV)}(x) = \frac{\partial^4 f}{\partial x^4}(x, y_m(x)) + 4\frac{\partial^4 f}{\partial x^3 \partial y}(x, y_m(x)) \cdot y_m'(x) + \frac{\partial^4 f}{\partial x^4}(x, y_m(x)) \cdot$$

$$\cdot [y_m'(x)]^4 + 4\frac{\partial^4 f}{\partial x \partial y^3}(x, y_m(x)) \cdot [y_m'(x)]^3 + 3\frac{\partial^4 f}{\partial x^3 \partial y}(x, y_m(x)) \cdot [y_m'(x)]^2 +$$

$$3\frac{\partial^4 f}{\partial x^2 \partial y^2}(x, y_m(x)) \cdot [y_m'(x)]^2 + 6\frac{\partial^3 f}{\partial x^2 \partial y}(x, y_m(x)) \cdot y_m''(x) + 12\frac{\partial^3 f}{\partial x \partial y^2}(x, y_m(x)) \cdot$$

$$\cdot y_m'(x) \cdot y_m''(x) + 6\frac{\partial^3 f}{\partial y^3}(x, y_m(x)) \cdot [y_m'(x)]^2 \cdot y_m''(x) + 4\frac{\partial^2 f}{\partial x \partial y}(x, y_m(x)) \cdot y_m'''(x) +$$

$$3\frac{\partial^2 f}{\partial y^2}(x, y_m(x)) \cdot [y_m''(x)]^2 + 4\frac{\partial^2 f}{\partial x^2}(x, y_m(x)) \cdot y_m'(x) \cdot y_m'''(x) + \frac{\partial f}{\partial y}(x, y_m(x)) \cdot y_m^{IV}(x)$$

and the following estimations holds

$$\left|F_m^{(IV)}(x)\right| \leq M_4(M+1)^4 + 6M_1M_3(M+1)^2 + 3M_2M_1^2(M+1)^2 + 4M_2^2(M+1)^3 +$$

$$+4M_2M_1^2(M+1)^2 + M_1M''' = M^{IV},\ \forall x \in [x_0, x_0 + h],\ \forall m \in \mathbb{N}.$$

Then, for the remainder from (34) we have the estimation,

$$|R_{m,i}(f)| \leq \frac{h^5 M^{IV}}{720n^4},\ \forall i = \overline{1,n}, \forall m \in \mathbb{N}^*.$$

So, we obtain in [6] the following result, which improve the results from [21], giving a concise error estimation :

Theorem 7 *(in [6]) :If* $f \in C^4([x_0 - a, x_0 + a] \times [y_0 - b, y_0 + b])$, *in the conditions of Theorem 5, the error estimation (28) became :*

$$\left| y^*(x_i) - \overline{y_m(x_i)} \right| \leq \frac{(Lh)^m}{1 - Lh}(|y_0| + Mh) +$$

$$+ \frac{h^5 M^{IV}}{720n^4(1 - [hL + \frac{h^2}{12n^2}(M_2(M+1) + M_1 L)])}, \ \forall i = \overline{1,n}, \forall m \in \mathbb{N}, m \geq 2, \tag{42}$$

$\forall i = \overline{1,n}$, $\forall m \in \mathbb{N}, m \geq 2$.

Remark 8 *The error estimations given in the Theorems 1-7 present the adaptability of the method of successive approximations, corresponding to the properties of the function* f.

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