THE SOLUTION OF AN OPEN QUESTION

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Abstract. In this short note, we obtain the solution of OQ.1277:
The inequality

$$I > \frac{\alpha A + \beta G}{\alpha + \beta} > \left(A^{\gamma} G^{\delta}\right)^{\frac{1}{\gamma+\delta}} > \sqrt{AG}$$

holding if and only if $1 < \frac{\gamma}{\delta} < \frac{\alpha}{\beta} < 2$ for $\alpha > \beta > 0$, and $\gamma > \delta > 0$, where $a > b > 0$, and

$$I = I(a,b) = \frac{1}{e}\left(\frac{a^a}{b^b}\right)^{\frac{1}{a-b}}, \; A = A(a,b) = \frac{a+b}{2}, \; G = G(a,b) = \sqrt{ab}.$$

1. Introduction

In [1], an interesting and open question is posed by Mihály Bencze. That is
OQ.1277. Determine all $\alpha > \beta > 0$ and $\gamma > \delta > 0$, such that

$$I > \frac{\alpha A + \beta G}{\alpha + \beta} > \left(A^{\gamma} G^{\delta}\right)^{\frac{1}{\gamma+\delta}} > \sqrt{AG}, \tag{1.1}$$

where $a > b > 0$, and

$$I = I(a,b) = \frac{1}{e}\left(\frac{a^a}{b^b}\right)^{\frac{1}{a-b}}, A = A(a,b) = \frac{a+b}{2}, G = G(a,b) = \sqrt{ab}. \tag{1.2}$$

In this short note, we obtain the solution of OQ.1277.

Theorem 1.1. *The inequalities* (1.1) *holding if and only if* $1 < \frac{\gamma}{\delta} < \frac{\alpha}{\beta} < 2$ *for* $\alpha > \beta > 0$, *and* $\gamma > \delta > 0$.

2. lemma

In order to prove Theorem1.1 above, we require some lemmas.

Lemma 2.1. *If* $x > y > 0$, *and* $\gamma > \delta > 0$. *We then have the inequality*

$$\left(x^{\gamma} y^{\delta}\right)^{\frac{1}{\gamma+\delta}} > \sqrt{xy}. \tag{2.1}$$

Lemma 2.2. *(* [2]*) If* $x > y > 0$, *and* $\frac{\alpha}{\beta} > \frac{\gamma}{\delta}$ *for* $\alpha > \beta > 0, \gamma > \delta > 0$. *Then the following inequality holds*

$$\frac{\alpha x + \beta y}{\alpha + \beta} > \left(x^{\gamma} y^{\delta}\right)^{\frac{1}{\gamma+\delta}}. \tag{2.2}$$

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Lemma 2.3. *If $x > y > 0$, and $\frac{\alpha}{\beta} < 2$ for $\alpha > \beta > 0$. We then have the inequality*

$$\frac{2x+y}{3} > \frac{\alpha x + \beta y}{\alpha + \beta}. \tag{2.3}$$

Lemma 2.4. *([2])Heinz-Seiffert's inequality*

$$I > \frac{2A+G}{3} \tag{2.4}$$

holds for any $a > 0$, and $b > 0$.

3. The Proof of Theorem 1.1

Now, we prove Theorem1.1.

Proof. Firstly, for $1 < \frac{\gamma}{\delta} < \frac{\alpha}{\beta} < 2$, $\alpha > \beta > 0$, and $\gamma > \delta > 0$, let $x = A, y = G$, and combining Lemma2.1-2.4, we obtain the inequalities (1.1).
Next, we prove that $1 < \frac{\gamma}{\delta} < \frac{\alpha}{\beta} < 2(\alpha > \beta > 0, \gamma > \delta > 0)$ are the best possible for (1.1). Assume the following inequalities have holden for any $x > 1$:

$$I(x,1) > \frac{\alpha A(x,1) + \beta G(x,1)}{\alpha + \beta} > \Big(A^{\gamma}(x,1)G^{\delta}(x,1)\Big)^{\frac{1}{\gamma+\delta}} > \sqrt{A(x,1)G(x,1)}. \tag{3.1}$$

There is no harm in supposing $1 < x \leqslant 2$ (In fact, if $n < x \leqslant n+1$, we can take $x = t+n$, where n is a positive integer). Setting $x = t+1$, applying Taylor's Theorem to the functions $G(x,1), A(x,1)$, and $I(x,1)$, we have

$$G(x,1) = G(t+1,1) = 1 + \frac{1}{2}t - \frac{1}{8}t^2 + \cdots, \tag{3.2}$$

$$A(x,1) = A(t+1,1) = 1 + \frac{1}{2}t, \tag{3.3}$$

$$I(x,1) = I(t+1,1) = 1 + \frac{1}{2}t - \frac{1}{24}t^2 + \cdots. \tag{3.4}$$

and

$$\frac{\alpha A(x,1) + \beta G(x,1)}{\alpha + \beta} = 1 + \frac{1}{2}t - \frac{\beta}{8(\alpha+\beta)}t^2 + \cdots, \tag{3.5}$$

$$\Big(A^{\gamma}(x,1)G^{\delta}(x,1)\Big)^{\frac{1}{\gamma+\delta}} = 1 + \frac{1}{2}t - \frac{\delta}{8(\gamma+\delta)}t^2 + \cdots, \tag{3.6}$$

$$\sqrt{A(x,1)G(x,1)} = 1 + \frac{1}{2}t - \frac{1}{16}t^2 + \cdots. \tag{3.7}$$

With simple manipulations (3.4)-(3.7), together with (3.1), yield

$$-\frac{1}{24} > -\frac{\beta}{8(\alpha+\beta)} > -\frac{\delta}{8(\gamma+\delta)} > -\frac{1}{16}. \tag{3.8}$$

From (3.8), it immediately follows that

$$1 < \frac{\gamma}{\delta} < \frac{\alpha}{\beta} < 2(\alpha > \beta > 0, \gamma > \delta > 0).$$

The proof of Theorem1.1 is completed. ∎

References

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