TWO WEIGHTED PRODUCT TYPE MEANS AND ITS MONOTONICITIES

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Abstract. In this paper, we define the following two weighted product type means in n variables and prove their monotonicities:

$$I_n^{[r]}(a,\lambda) = \prod_{\substack{i_0+i_1+\cdots+i_n=n+r \\ i_0,i_1,\ldots,i_n\geq 1 \text{ are integers}}} \left[\sum_{k=0}^{n} \frac{i_k}{n+r} a_k\right]^{\frac{\sum_{k=0}^{n}(i_k-1)\lambda_k}{\binom{n+r-1}{r-2}\sum_{i=0}^{n}\lambda_i}},$$

$$I_n^{*[r]}(a,\lambda) = \prod_{\substack{i_0+i_1+\cdots+i_n=r \\ i_0,i_1,\ldots,i_n\geq 0 \text{ are integers}}} \left[\sum_{k=0}^{n} \frac{i_k}{r} a_k\right]^{\frac{\sum_{k=0}^{n}(1+i_k)\lambda_k}{\binom{n+r+1}{r}\sum_{i=0}^{n}\lambda_i}},$$

where $a = (a_0, a_1, \cdots, a_n), \lambda = (\lambda_0, \lambda_1, \cdots, \lambda_n)$ $a_i \geqslant 0$ and $\lambda_i > 0$ for $0 \leq i \leq n$ and r be a nonnegative integer.

1. Introduction

For positive numbers a_0, a_1, let

$$I = I(a_0,a_1) = \begin{cases} \exp\left[\dfrac{a_1 \ln a_1 - a_0 \ln a_0}{a_1 - a_0} - 1\right], & a_0 < a_1, \\ a_0, & a_0 = a_1. \end{cases} \tag{1.1}$$

This is called the identric mean (see [1]).

In 2003, Xiao and Zhang [2] gave a new product type mean and its dual form in two variables respectively as follows

$$I(a_0,a_1;k) = \prod_{i=1}^{k} \left(\frac{(k+1-i)a_0 + i a_1}{k+1}\right)^{\frac{1}{k}}, \tag{1.2}$$

and

$$I^*(a_0,a_1;k) = \prod_{i=0}^{k} \left(\frac{(k-i)a_0 + i a_1}{k}\right)^{\frac{1}{k+1}}, \tag{1.3}$$

where k is a natural number. Authors proved that $I(a_0, a_1; k)$ is a monotone decreasing function and $I'(a_0, a_1; k)$ is a monotone increasing function with k, and

$$\lim_{k\to+\infty} I(a_0,a_1;k) = \lim_{k\to+\infty} I^*(a_0,a_1;k) = I(a_0,a_1).$$

Let $a = (a_0, a_1, \cdots, a_n)$ and r be a nonnegative integer, where a_i for $0 \leq i \leq n$ are nonnegative real numbers. Then

$$I_n^{[r]}(a) = \prod_{\substack{i_0+i_1+\cdots+i_n=n+r \\ i_0,i_1,\ldots,i_n\geq 1}} \left[\sum_{k=0}^{n} \frac{i_k}{n+r} a_k\right]^{\frac{1}{\binom{n+r-1}{r-1}}}, \tag{1.4}$$

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and

$$I_n^{*[r]}(a) = \prod_{\substack{i_0+i_1+\cdots+i_n=r \\ i_0,i_1,\ldots,i_n\geq 0}} \left[\sum_{k=0}^{n} \frac{i_k}{r} a_k\right]^{\frac{1}{\binom{n+r}{r}}}, \tag{1.5}$$

are called the product type mean and its dual form of a, respectively.

Also, in 2003, Zhang and Xiao [2] obtained that for any nonnegative integers r, s with $s > r$, then

$$I_n^{[r]}(a) \geqslant I_n^{[s]}(a), \tag{1.6}$$

and

$$I_n^{*[r]}(a) \leqslant I_n^{*[s]}(a), \tag{1.7}$$

with equalities if and only if $a_0 = a_1 = \cdots = a_n$, and

$$\lim_{r\to\infty} I_n^{[r]}(a) = \lim_{r\to\infty} I_n^{*[r]}(a) = I(a) = \exp\left\{\frac{V(a;n,1)}{V(a;n,0)} - \sum_{k=1}^{n}\frac{1}{k}\right\}, \tag{1.8}$$

where $I(a)$ is called the identric mean in n variables, and $a_i \neq a_j$ for $i \neq j$,

$$V(a;r,k) = \begin{vmatrix} 1 & a_0 & a_0^2 & \cdots & a_0^{n-1} & a_0^r \ln^k a_0 \\ 1 & a_1 & a_1^2 & \cdots & a_1^{n-1} & a_1^r \ln^k a_1 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & a_n & a_n^2 & \cdots & a_n^{n-1} & a_n^r \ln^k a_n \end{vmatrix}. \tag{1.9}$$

In this paper, the definition of the weighted product type mean and its dual form in n variables are given, its monotonicities are obtained.

2. Main Results

Definition 2.1. *Let $a = (a_0, a_1, \cdots, a_n), \lambda = (\lambda_0, \lambda_1, \cdots, \lambda_n)$ and r be a nonnegative integer, where $a_i \geqslant 0$ and $\lambda_i > 0$ for $0 \leq i \leq n$, then*

$$I_n^{[r]}(a,\lambda) = \prod_{\substack{i_0+i_1+\cdots+i_n=n+r \\ i_0,i_1,\ldots,i_n\geq 1 \text{ are integers}}} \left[\sum_{k=0}^{n} \frac{i_k}{n+r} a_k\right]^{\frac{\sum_{k=0}^{n}(i_k-1)\lambda_k}{\binom{n+r-1}{r-2}\sum_{i=0}^{n}\lambda_i}}, \tag{2.1}$$

and

$$I_n^{*[r]}(a,\lambda) = \prod_{\substack{i_0+i_1+\cdots+i_n=r \\ i_0,i_1,\ldots,i_n\geq 0 \text{ are integers}}} \left[\sum_{k=0}^{n} \frac{i_k}{r} a_k\right]^{\frac{\sum_{k=0}^{n}(1+i_k)\lambda_k}{\binom{n+r+1}{r}\sum_{i=0}^{n}\lambda_i}}, \tag{2.2}$$

is called the weighted product type mean and its dual form of a for λ, respectively.

Now, we give a theorem relating to the weighted product type mean $I_n^{[r]}(a,\lambda)$ and its dual form $I_n^{*[r]}(a,\lambda)$.

Theorem 2.1. *If $r \in N$, then $I_n^{[r]}(a,\lambda)$ is a monotone decreasing function and $I_n^{*[r]}(a,\lambda)$ is a monotone increasing function with r, those are*

$$I_n^{[r]}(a,\lambda) \geqslant I_n^{[r+1]}(a,\lambda) \geqslant I(a,\lambda) \geqslant I_n^{*[r+1]}(a,\lambda) \geqslant I_n^{*[r]}(a,\lambda), \tag{2.3}$$

with equality holding if and only if $a_0 = a_1 = \cdots = a_n$, where

$$\lim_{r\to\infty} I_n^{[r]}(a,\lambda) = \lim_{r\to\infty} I_n^{*[r]}(a,\lambda) = I(a,\lambda) = \exp\left\{\frac{\int_E \left(\sum_{i=0}^{n}\lambda_i x_i\right)\ln\left(\sum_{i=0}^{n} a_i x_i\right)dx}{\int_E \left(\sum_{i=0}^{n}\lambda_i x_i\right)dx}\right\}, \tag{2.4}$$

$x_0 = 1 - \sum_{i=1}^{n} x_i$, *and* $dx = dx_1 dx_2 \cdots dx_n$ *denotes the differential of the volume in* E*:*

$$E = \left\{ (x_1, x_2, \cdots, x_n) : \sum_{i=1}^{n} x_i \leqslant 1, x_i \geqslant 0, i = 1, 2, \cdots, n \right\}. \tag{2.5}$$

Proof. Firstly, we prove that $I_n^{[r]}(a, \lambda)$ is a monotone decreasing function with r, i.e. the following inequality holds:

$$I_n^{[r]}(a, \lambda) \geqslant I_n^{[r+1]}(a, \lambda), \tag{2.6}$$

or

$$\prod_{\substack{i_0+i_1+\cdots+i_n=n+r \\ i_0,i_1,\ldots,i_n \geq 1 \text{ are integers}}} \left[\sum_{k=0}^{n} \frac{i_k}{n+r} a_k \right]^{\frac{\sum_{k=0}^{n}(i_k-1)\lambda_k}{\binom{n+r-1}{r-2} \sum_{i=0}^{n} \lambda_i}} \tag{2.7}$$

$$\geqslant \prod_{\substack{i_0+i_1+\cdots+i_n=n+r+1 \\ i_0,i_1,\ldots,i_n \geq 1 \text{ are integers}}} \left[\sum_{k=0}^{n} \frac{i_k}{n+r+1} a_k \right]^{\frac{\sum_{k=0}^{n}(i_k-1)\lambda_k}{\binom{n+r}{r-1} \sum_{i=0}^{n} \lambda_i}}.$$

If $r \in N$ and $i_0 + i_1 + i_2 + \cdots + i_n = n + r$, then

$$\begin{aligned}
\sum_{j=0}^{n} (\nu_j - 1) \left[\sum_{k=0}^{n} (\nu_k - 1)\lambda_k - \lambda_j \right] &= \sum_{j=0}^{n} (\nu_j - 1) \sum_{k=0}^{n} (\nu_k - 1)\lambda_k - \sum_{j=0}^{n} (\nu_j - 1)\lambda_j \\
&= r \sum_{k=0}^{n} (\nu_k - 1)\lambda_k - \sum_{k=0}^{n} (\nu_k - 1)\lambda_k \\
&= (r-1) \sum_{k=0}^{n} (\nu_k - 1)\lambda_k,
\end{aligned}$$

and

$$\begin{aligned}
\sum_{k=0}^{n} \frac{i_k}{n+r} a_k &= \sum_{k=0}^{n} \frac{i_k}{n+r+1} \cdot \frac{n+r+1}{n+r} a_k = \frac{n+r+1}{n+r} \sum_{k=0}^{n} \frac{i_k}{n+r+1} a_k \\
&= \frac{1}{n+r} \left[\sum_{j=0}^{n} i_j + 1 \right] \sum_{k=0}^{n} \frac{i_k}{n+r+1} a_k \\
&= \frac{1}{n+r} \left[\sum_{j=0}^{n} i_j \sum_{k=0}^{n} \frac{i_k}{n+r+1} a_k + \sum_{k=0}^{n} \frac{i_k}{n+r+1} a_k \right] \\
&= \frac{1}{n+r} \left[\sum_{j=0}^{n} i_j \sum_{k=0}^{n} \frac{i_k}{n+r+1} a_k + \sum_{j=0}^{n} \frac{i_j}{n+r+1} a_j \right] \\
&= \sum_{j=0}^{n} \frac{i_j}{n+r} \left[\sum_{k=0}^{n} \frac{i_k a_k}{n+r+1} + \frac{a_j}{n+r+1} \right].
\end{aligned}$$

By using the weighted arithmetic-geometric mean inequality, we have

$$\sum_{k=0}^{n} \frac{i_k}{n+r} a_k \geq \prod_{j=0}^{n} \left[\sum_{k=0}^{n} \frac{i_k a_k}{n+r+1} + \frac{a_j}{n+r+1} \right]^{i_j/(n+r)},$$

and then

$$
\begin{aligned}
&\prod_{\substack{i_0+i_1+\cdots+i_n=n+r\\ i_0,i_1,\ldots,i_n\geq 1}} \left[\sum_{k=0}^{n} \frac{i_k}{n+r} a_k\right]^{\sum_{k=0}^{n}(i_k-1)\lambda_k} \\
&\geq \prod_{\substack{i_0+i_1+\cdots+i_n=n+r\\ i_0,i_1,\ldots,i_n\geq 1}} \left\{\prod_{j=0}^{n}\left[\sum_{k=0}^{n} \frac{i_k a_k}{n+r+1} + \frac{a_j}{n+r+1}\right]^{i_j/(n+r)}\right\}^{\sum_{k=0}^{n}(i_k-1)\lambda_k} \\
&= \prod_{j=0}^{n} \prod_{\substack{i_0+i_1+\cdots+i_n=n+r\\ i_0,i_1,\ldots,i_n\geq 1}} \left[\sum_{k=0}^{n} \frac{i_k a_k}{n+r+1} + \frac{a_j}{n+r+1}\right]^{i_j\sum_{k=0}^{n}(i_k-1)\lambda_k/(n+r)} \\
&= \prod_{j=0}^{n} \prod_{\substack{\nu_0+\nu_1+\cdots+\nu_n=n+r+1\\ \nu_0,\nu_1,\ldots,\nu_{j-1},\nu_{j+1},\ldots,\nu_n\geq 1;\nu_j\geq 2}} \left[\sum_{k=0}^{n} \frac{\nu_k}{n+r+1} a_k\right]^{(\nu_j-1)[\sum_{k=0}^{n}(\nu_k-1)\lambda_k+\lambda_j]/(n+r)} \qquad (2.8)\\
&= \prod_{\substack{\nu_0+\nu_1+\cdots+\nu_n=n+r+1\\ \nu_0,\nu_1,\ldots,\nu_n\geq 1}} \left[\sum_{k=0}^{n} \frac{\nu_k}{n+r+1} a_k\right]^{\sum_{j=0}^{n}(\nu_j-1)[\sum_{k=0}^{n}(\nu_k-1)\lambda_k+\lambda_j]/(n+r)} \\
&= \prod_{\substack{\nu_0+\nu_1+\cdots+\nu_n=n+r+1\\ \nu_0,\nu_1,\ldots,\nu_n\geq 1}} \left[\sum_{k=0}^{n} \frac{\nu_k}{n+r+1} a_k\right]^{(r-1)\sum_{j=0}^{n}(\nu_j-1)\lambda_j\Big/(n+r)} \\
&= \prod_{\substack{\nu_0+\nu_1+\cdots+\nu_n=n+r+1\\ \nu_0,\nu_1,\ldots,\nu_n\geq 1}} \left[\sum_{k=0}^{n} \frac{\nu_k}{n+r+1} a_k\right]^{\sum_{j=0}^{n}(\nu_j-1)\lambda_j\binom{n+r-2}{r-1}\Big/\binom{n+r}{r}},
\end{aligned}
$$

notice that the result from line 4 to line 5 in (2.8) follows from a simple fact that

$$\left[\sum_{k=0}^{n} \frac{\nu_k}{n+r} a_k\right]^{(\nu_j-1)[\sum_{k=0}^{n}(\nu_k-1)\lambda_k+\lambda_j]/(n+r)} = 1$$

for $\nu_j = 1$. The equalities above are valid if and only if $\sum_{k=0}^{n} \frac{i_k a_k}{n+r} + \frac{a_0}{n+r} = \sum_{k=0}^{n} \frac{i_k a_k}{n+r} + \frac{a_1}{n+r} = \cdots = \sum_{k=0}^{n} \frac{i_k a_k}{n+r} + \frac{a_n}{n+r}$ which is equivalent to $a_0 = a_1 = \cdots = a_n$. This implies that inequality (2.7) or (2.6).

Secondly, we can similarly obtain that $I_n^{*[r]}(a,\lambda)$ is a monotone increasing function with r.

In the final, it is easy to see that $\lim_{r\to\infty} I_n^{[r]}(a,\lambda) = \lim_{r\to\infty} I_n^{*[r]}(a,\lambda)$, and straightforward computation yields

$$
\begin{aligned}
\ln \lim_{r\to\infty} I_n^{*[r]}(a,\lambda) &= \lim_{r\to\infty} \ln I_n^{*[r]}(a,\lambda) \\
&= \lim_{r\to\infty} \frac{\sum_{k=0}^{n}(1+i_k)\lambda_k}{\binom{n+r+1}{r}\sum_{i=0}^{n}\lambda_i} \sum_{\substack{i_0+i_1+\cdots+i_n=r\\ i_0,i_1,\ldots,i_n\geq 0}} \ln \sum_{k=0}^{n} \frac{i_k}{r} a_k \\
&= \frac{\int_E \left(\sum_{i=0}^{n}\lambda_i x_i\right)\ln(\sum_{i=0}^{n} a_i x_i)dx}{\int_E \left(\sum_{i=0}^{n}\lambda_i x_i\right)dx} \\
&= \ln I(a,\lambda),
\end{aligned}
$$

where $x_0 = 1 - \sum_{i=1}^{n} x_i$, and $dx = dx_1 dx_2 \cdots dx_n$ denotes the differential of the volume in E as (2.5).

The proof of Theorem 2.1 is completed. $\square$

References

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