COMPLETELY MONOTONIC FUNCTIONS RELATED TO THE GAMMA FUNCTIONS

CHAO-PING CHEN AND FENG QI

ABSTRACT. (i) Let $a, b > 0$ be real numbers, and let

$$f_{a,b}(x) = \frac{1}{x^{b-a}} \left[\frac{\Gamma(bx+1)}{\Gamma(ax+1)} \right]^{1/x}.$$

Then, for $x > 0$ and $n = 1, 2, \ldots$, $(-1)^n (\ln f_{a,b}(x))^{(n)} \gtreqless 0$ according as $b \gtreqless a$.

(ii) Let $p > 0$ be a real number, and let $f_p(x) = \theta(px) - p\theta(x)$, where

$$\theta(x) = \int_0^\infty \left(\frac{t}{e^t - 1} - 1 + \frac{t}{2} \right) \frac{e^{-xt}}{t^2} \, \mathrm{d}t, x > 0$$

is remainder of Binet's formula. Then, for $x > 0$ and $n = 0, 1, 2, \ldots$,

$$(-1)^n f_p^{(n)}(x) \gtreqless 0 \quad \text{according as} \quad p \lesseqgtr 1.$$

1. Introduction

The Euler gamma function Γ and its logarithmic derivative ψ, the so-called digamma function, are defined for $\operatorname{Re} z > 0$ by

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} \, \mathrm{d}t \quad \text{and} \quad \psi(z) = \frac{\Gamma'(z)}{\Gamma(z)}.$$

There exists a very extensive literature on these functions. In particular, inequalities, monotonicity and complete monotonicity properties for these functions have been published, we refer to the paper [1] and [2], and the references given therein. We recall that a function f is said to be completely monotonic on an interval I, if f has derivatives of all orders on I and satisfies

$$(-1)^n f^{(n)}(x) \geq 0 \quad (x \in I; n = 0, 1, 2, \ldots). \tag{1}$$

If the inequality (1) is strict, then f is said to be strictly completely monotonic on I. Completely monotonic functions have remarkable applications in different branches. For instance, they play a role in potential theory [4], probability theory [6, 8, 10], physics [7], numerical and asymptotic analysis [9, 15], and combinatorics [3]. A detailed collection of the most important properties of completely monotonic functions can be found in [14, Chapter IV], and in an abstract in [5].

In a recent paper [12], the terminology "(strictly) logarithmically completely monotonic function" was introduced. It was also shown in this paper that a (strictly) logarithmically completely monotonic function is also (strictly) completely

2000 *Mathematics Subject Classification.* 33B15; 26A48.

Key words and phrases. Gamma function, psi function, complete monotonicity, logarithmically complete monotonicity, Binet's formula, remainder.

The authors were supported in part by SF of Henan Innovation Talents at Universities, China.

monotonic. For convenience, we recall that a positive function f is said to be logarithmically completely monotonic on an interval I if its logarithm $\ln f$ satisfies

$$(-1)^n(\ln f(x))^{(n)} \geq 0 \quad (x \in I; n = 1, 2, \ldots). \tag{2}$$

If inequality (2) is strict, then f is said to be strictly logarithmically completely monotonic.

In 2003, J. Sándor [13] showed that

$$\lim_{x\to\infty} \frac{1}{x^{b-a}} \left[\frac{\Gamma(bx+1)}{\Gamma(ax+1)}\right]^{1/x} = \frac{b^b}{a^a} e^{b-a}. \tag{3}$$

Our first theorem considers logarithmically complete monotonicity property of the function in (3).

Theorem 1. *Let $a, b > 0$ be real numbers, and let*

$$f_{a,b}(x) = \frac{1}{x^{b-a}} \left[\frac{\Gamma(bx+1)}{\Gamma(ax+1)}\right]^{1/x}.$$

Then, for $x > 0$ and $n = 1, 2, \ldots$, $(-1)^n(\ln f_{a,b}(x))^{(n)} \gtreqless 0$ according as $b \gtreqless a$.

If we denote by

$$I(a, b) = \frac{1}{e}\left(\frac{b^b}{a^a}\right)^{1/(b-a)}, \quad a > 0, b > 0, a \neq b,$$

the so-called identric mean, then, we yield from (3) and the monotonicity of the function $f_{a,b}$ that, for $x > 0$,

$$\frac{1}{x^{b-a}} \left[\frac{\Gamma(bx+1)}{\Gamma(ax+1)}\right]^{1/x} \gtreqless [e^2 I(a, b)]^{b-a} \quad \text{according as} \quad b \gtreqless a. \tag{4}$$

Binet's formula [16, p. 103] states that for $x > 0$,

$$\ln\Gamma(x) = \left(x - \frac{1}{2}\right)\ln x - x + \ln\sqrt{2\pi} + \theta(x),$$

where

$$\theta(x) = \int_0^\infty \left(\frac{t}{e^t - 1} - 1 + \frac{t}{2}\right)\frac{e^{-xt}}{t^2}\,\mathrm{d}t. \tag{5}$$

Let $p > 0$ be a real number. Our second theorem considers complete monotonicity property of the function $x \mapsto \theta(px) - p\theta(x)$ on $(0, \infty)$.

Theorem 2. *Let $p > 0$ be a real number, and let $f_p(x) = \theta(px) - p\theta(x)$, where $\theta(x)$ is defined by (5). Then, for $x > 0$ and $n = 0, 1, 2, \ldots$,*

$$(-1)^n f_p^{(n)}(x) \gtreqless 0 \quad \text{according as} \quad p \lesseqgtr 1.$$

2. Proofs of Theorems

Proof of Theorem 1. Using Leibniz' rule

$$[u(x)v(x)]^{(n)} = \sum_{k=0}^{n} \binom{n}{k} u^{(k)}(x)v^{(n-k)}(x),$$

we obtain

$$(\ln f_{a,b}(x))^{(n)} = -\frac{(b-a)(-1)^{n-1}(n-1)!}{x^n}$$

$$
\begin{aligned}
&+\sum_{k=0}^{n}\binom{n}{k}\left(\frac{1}{x}\right)^{(n-k)}[\ln\Gamma(bx+1)-\ln\Gamma(ax+1)]^{(k)}\\
&=-\frac{(b-a)(-1)^{n-1}(n-1)!}{x^n}+\frac{(-1)^n n!}{x^{n+1}}[\ln\Gamma(bx+1)-\ln\Gamma(ax+1)]\\
&+\frac{(-1)^n n!}{x^{n+1}}\sum_{k=1}^{n}\frac{(-1)^k}{k!}x^k[b^k\psi^{(k-1)}(bx+1)-a^k\psi^{(k-1)}(ax+1)].
\end{aligned}
$$

Define for $x>0$,

$$
\begin{aligned}
g_{a,b}(x)&=\frac{(-1)^n x^{n+1}}{n!}(\ln f(x))^{(n)}\\
&=\frac{(b-a)x}{n}+\ln\Gamma(bx+1)-\ln\Gamma(ax+1)\\
&\quad+\sum_{k=1}^{n}\frac{(-1)^k}{k!}x^k[b^k\psi^{(k-1)}(bx+1)-a^k\psi^{(k-1)}(ax+1)].
\end{aligned}
$$

Using the representations

$$
\begin{aligned}
\frac{(n-1)!}{x^n}&=\int_0^\infty t^{n-1}e^{-xt}\,\mathrm{d}t,\,(x>0),\\
\psi^{(n)}(x)&=(-1)^{n+1}\int_0^\infty \frac{t^n}{1-e^{-t}}e^{-xt}\,\mathrm{d}t,\,(x>0,n=1,2,\dots),
\end{aligned}
$$

see [11, p. 16], we imply

$$
\begin{aligned}
\frac{n!}{x^n}g'_{a,b}(x)&=\frac{(b-a)(n-1)!}{x^n}+(-1)^n[b^{n+1}\psi^{(n)}(bx+1)-a^{n+1}\psi^{(n)}(ax+1)]\\
&=(b-a)\int_0^\infty t^{n-1}e^{-xt}\,\mathrm{d}t-\int_0^\infty\frac{b^{n+1}t^n}{e^t-1}e^{-bxt}\,\mathrm{d}t+\int_0^\infty\frac{a^{n+1}t^n}{e^t-1}e^{-axt}\,\mathrm{d}t\\
&=(b-a)\int_0^\infty t^{n-1}e^{-xt}\,\mathrm{d}t-\int_0^\infty\frac{t^n}{e^{t/b}-1}e^{-xt}\,\mathrm{d}t+\int_0^\infty\frac{t^n}{e^{t/a}-1}e^{-xt}\,\mathrm{d}t\\
&=\int_0^\infty\left[\left(\frac{t}{e^{t/a}-1}-a\right)-\left(\frac{t}{e^{t/b}-1}-b\right)\right]t^{n-1}e^{-xt}\,\mathrm{d}t.
\end{aligned}
$$

For fixed $t>0$, we define the function

$$h_t(a)=\frac{t}{e^{t/a}-1}-a\quad(a>0).$$

Differentiation yields

$$h'_t(a)=\frac{(t/a)^2e^{t/a}-(e^{t/a}-1)^2}{(e^{t/a}-1)^2}.$$

Now we are in a position to prove $h'_t(a)<0$ for $a>0$, which is equivalent to

$$(t/a)e^{t/(2a)}<e^{t/a}-1,$$

i.e.,

$$(t/a)<e^{t/(2a)}-e^{-t/(2a)}.$$

Using power series expansion, we have

$$e^{t/(2a)} - e^{-t/(2a)} - (t/a) = 2\sum_{n=2}^{\infty} \frac{1}{(2n-1)!}\left(\frac{t}{2a}\right)^{2n-1} > 0$$

for $a > 0$. Hence $h_t'(a) < 0$ for $a > 0$, and then, for $x > 0$, $g_{a,b}'(x) \gtreqless 0$ and $g_{a,b}(x) \gtreqless g_{a,b}(0) = 0$ according as $b \gtreqless a$. This implies that for $x > 0$ and $n = 1, 2, \dots$, $(-1)^n(\ln f_{a,b}(x))^{(n)} \gtreqless 0$ according as $b \gtreqless a$. The proof is complete. □

Proof of Theorem 2. By (5), we imply

$$\begin{aligned}
f_p(x) &= \int_0^\infty \left(\frac{u}{e^u-1} - 1 + \frac{u}{2}\right)\frac{e^{-pxu}}{u^2}\,\mathrm{d}u - p\int_0^\infty \left(\frac{t}{e^t-1} - 1 + \frac{t}{2}\right)\frac{e^{-xt}}{t^2}\,\mathrm{d}t \\
&= p\int_0^\infty \left[\frac{t}{p(e^{t/p}-1)} - 1 + \frac{t}{2p}\right]\frac{e^{-xt}}{t^2}\,\mathrm{d}t - p\int_0^\infty \left(\frac{t}{e^t-1} - 1 + \frac{t}{2}\right)\frac{e^{-xt}}{t^2}\,\mathrm{d}t \\
&= p\int_0^\infty \left[\frac{t}{p(e^{t/p}-1)} - \frac{1}{e^t-1} + \frac{1-p}{p}\right]\frac{e^{-xt}}{t^2}\,\mathrm{d}t \\
&= \int_0^\infty \frac{\delta_p(t)}{2(e^{t/p}-1)(e^t-1)t}e^{-xt}\,\mathrm{d}t
\end{aligned}$$

and therefore,

$$(-1)^n f_p^{(n)}(x) = \int_0^\infty \frac{t^{n-1}\delta_p(t)}{2(e^{t/p}-1)(e^t-1)}e^{-xt}\,\mathrm{d}t.$$

where

$$\begin{aligned}
\delta_p(t) &= (1+p)e^t - (1+p)e^{t/p} + (1-p)e^{[(1+p)/p]t} + p - 1 \\
&= \sum_{k=3}^{\infty}[p^k - 1 + (1-p)(1+p)^{k-1}]\frac{(1+p)t^k}{p^k\cdot k!}.
\end{aligned}$$

It is easy to see that

$$p^k - 1 + (1-p)(1+p)^{k-1} = (p-1)\sum_{m=0}^{k-1} p^m + (1-p)\sum_{m=0}^{k-1}\binom{k-1}{m}p^m$$

$$= (p-1)\sum_{m=1}^{k-2}\left[1 - \binom{k-1}{m}\right]p^m \gtreqless 0 \quad \text{according as} \quad p \lesseqgtr 1.$$

This implies for $x > 0$ and $n \geq 0$,

$$(-1)^n f_p^{(n)}(x) \gtreqless 0 \quad \text{according as} \quad p \lesseqgtr 1.$$

The proof is complete. □

References

[1] H. Alzer, *Some gamma function inequalities*, Math. Comp. **60** (1993), 337–346.

[2] H. Alzer, *On some inequalities for the gamma and psi functions*, Math. Comp. **66** (1997), 373–389.

[3] K. Ball, *Completely monotonic rational functions and Hall's marriage theorem*, J. Comb. Th., Ser. B **61** (1994), 118–124.

[4] C. Berg, G. Forst, Potential Theory on Locally Compact Abelian Groups, Ergebnisse der Math. **87**, Springer, Berlin, 1975.

[5] C. Berg, J. P. R. Christensen, P. Ressel, Harmonic Analysis on Semigroups. Theory of Positive Definite and related Functions, raduate Texts in Mathematics **100**, Springer, Berlin-Heidelberg-New York, 1984.

[6] L. Bondesson, Generalized Gamma Convolutions and related Classes of Distributions and Densities, Lecture Notes in Statistics **76**, Springer, New York, 1992.

[7] W. A. Day, *On monotonicity of the relaxation functions of viscoelastic meterial*, Proc.Cambridge Philos. Soc. **67** (1970), 503–508.

[8] W. Feller, An Introduction to Probability Theory and its Applications, Vol. 2, Wiley, New York, 1966.

[9] C. L. Frenzen, *Error bounds for asymptotic expansions of the ratio of two gamma functions*, SIAM J. Math. Anal. **18** (1987), no. 3, 890–896.

[10] C. H. Kimberling, *A probabilistic interpretation of complete monotonicity*, Aequat. Math. **10** (1974), 152–164.

[11] W. Magnus, F. Oberhettinger, and R. P. Soni, Formulas and theorems for the special functions of mathematical physics, Springer, Berlin, 1966.

[12] F. Qi and Ch.-P. Chen, *A complete monotonicity property of the gamma function*, J. Math. Anal. Appl. **296** (2004), no. 2, 603–607.

[13] J. Sándor, *On a limit involving the Euler-gamma function*, Octogon Mathematical Magazine, **11** (2004), no. 1, 262–264.

[14] D. V. Widder, The Laplace Transform, Princeton Univ. Press, Princeton, NJ, 1941.

[15] J. Wimp, Sequence Transformations and their Applications, Academic Press, Nex York, 1981.

[16] Zh.-X. Wang and D.-R. Guo, Introduction to Special Function, The Series of Advanced Physics of Peking University, Peking University Press, Beijing, China, 2000. (Chinese)

(Ch.-P. Chen) Department of Applied Mathematics and Informatics, Research Institute of Applied Mathematics, Henan Polytechnic University, Jiaozuo City, Henan 454010, China

E-mail address: chenchaoping@hpu.edu.cn, chenchaoping@sohu.com

(F. Qi) Department of Applied Mathematics and Informatics, Research Institute of Applied Mathematics, Henan Polytechnic University, Jiaozuo City, Henan 454010, China

E-mail address: qifeng@hpu.edu.cn, fengqi618@member.ams.org

URL: http://rgmia.vu.edu.au/qi.html