About Surányi's Inequality

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Abstract. In the Miklós Schweitzer Mathematical Competition (Hungary) Professor János Surányi proposed the following problem, which is interesting and presents an aspect of a theorem. In this paper we present a new demonstration, some interesting applications and a generalization.

Theorem 1. (János Surányi). *If* $x_k > 0$ $(k = 1, 2, ..., n)$ *then the following inequality holds:*

$$(n-1)\sum_{k=1}^{n} x_k^n + n\prod_{k=1}^{n} x_k \geq \left(\sum_{k=1}^{n} x_k\right)\left(\sum_{k=1}^{n} x_k^{n-1}\right).$$

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Proof. Using mathematical induction, for $n = 2$ we obtain $x_1^2 + x_2^2 + 2x_1x_2 \geq (x_1 + x_2)^2$, which is true.

We suppose that is true for n and we prove for $n + 1$.

Because the inequality is symmetric and homogeneous we can suppose that $x_1 \geq x_2 \geq ... \geq x_{n+1}$ and $x_1 + x_2 + ... + x_n = 1$, so we must prove the following inequality:

$$n\sum_{k=1}^{n+1} x_k^{n+1} + (n+1)\prod_{k=1}^{n+1} x_k \geq \left(\sum_{k=1}^{n+1} x_k\right)\left(\sum_{k=1}^{n+1} x_k^n\right)$$

which can be written in the form

$$n\sum_{k=1}^{n} x_k^{n+1} + nx_{n+1}^{n+1} + nx_{n+1}\prod_{k=1}^{n} x_k + x_{n+1}\prod_{k=1}^{n} x_k - (1 + x_{n+1})\left(\sum_{k=1}^{n} x_k^n + x_{n+1}^n\right) \geq 0$$

From the inductive condition holds

$$nx_{n+1}\prod_{k=1}^{n} x_k \geq x_{n+1}\sum_{k=1}^{n} x_k^{n-1} - (n-1)\,x_{n+1}\sum_{k=1}^{n} x_k^n$$

It remains to prove that:

$$\left(n\sum_{k=1}^{n} x_k^{n+1} - \sum_{k=1}^{n} x_k^n\right) - x_{n+1}\left(n\sum_{k=1}^{n} x_k^n - \sum_{k=1}^{n} x_k^{n-1}\right) + x_{n+1}\left(\prod_{k=1}^{n} x_k + (n-1)\,x_{n+1}^n - x_{n+1}^{n-1}\right) \geq 0,$$

but this inequality can be decomposed in two inequalities in the following manner:

First, from the Chebyshev inequality we have:

$$n\sum_{k=1}^{n} x_k^n - \sum_{k=1}^{n} x_k^{n-1} \geq 0.$$

Second, because

$$nx_k^{n+1} + \frac{1}{n}x_k^{n-1} \geq 2x_k^n \quad (k = 1, 2, ..., n),$$

then after addition we have:

$$\begin{aligned}
&\prod_{k=1}^{n} x_k + (n-1)\, x_{n+1}^{n} - x_{n+1}^{n-1} \\
&= \prod_{k=1}^{n} (x_k - x_{n+1} + x_{n+1}) + (n-1)\, x_{n+1}^{n} - x_{n+1}^{n-1} \\
&\geq x_{n+1}^{n} + x_{n+1}^{n-1} \sum_{k=1}^{n} (x_k - x_{n+1}) + (n-1)\, x_{n+1}^{n} - x_{n+1}^{n-1} = 0
\end{aligned}$$

or

$$n \sum_{k=1}^{n} x_k^{n+1} - \sum_{k=1}^{n} x_k^{n} \geq \frac{1}{n} \left(n \sum_{k=1}^{n} x_k^{n} - \sum_{k=1}^{n} x_k^{n-1} \right),$$

but from $x_{n+1} \leq \frac{1}{n}$ holds the desired inequality.

If in Theorem 1 we take $n = 3$, then we obtain:.

Application 1. If $x_1, x_2, x_3 \geq 0$, then

$$x_1^3 + x_2^3 + x_3^3 + 3x_1x_2x_3 \geq x_1^2\,(x_2 + x_3) + x_2^2(x_3 + x_1) + x_3^2\,(x_1 + x_2)$$

which is the well known Schur's inequality. Therefore, the inequality of Surányi has generalized the Schur inequality.

Application 2. If a, b, c denote the sides of triangle ABC, s the semiperimeter, R the radius of the circumcircle, r the radius of the incircle, then:.

1). $R \geq 2r$ (the inequality of Euler)

2). $s^2 \geq r^2 + 16Rr$

3). $(4R + r)^3 \geq s^2\,(16R - 5r)$.

Proof. In Application 1 we take:

1). $x_1 = a, x_2 = b, x_3 = c$

2). $x_1 = s - a, x_2 = s - b, x_3 = s - c$

3). $x_1 = r_a, x_2 = r_b, x_3 = r_c$

where r_a, r_b, r_c are the radii of exinscribed circles.

If In Theorem 1 we take $n = 4$, then we obtain the following:.

Application 3. If $x_1, x_2, x_3, x_4 \geq 0$, then

$$2 \left(\sum_{k=1}^{4} x_k^4 + 2 \prod_{k=1}^{4} x_k \right) \geq \sum_{1 \leq i < j \leq 4} x_i x_j \left(x_i^2 + x_j^2 \right)$$

.

Remark. Because $x_i^2 + x_j^2 \geq 2x_ix_j$, then

$$\sum_{k=1}^{4} x_k^4 + 2 \prod_{k=1}^{4} x_k \geq \sum_{1 \leq i < j \leq 4} x_i^2 x_j^2,$$

but this is the Turkevici inequality. Therefore the inequality of Surányi gives a refinement and a generalization of Turkevici‘s inequality..

Application 4. Denote r_a, r_b, r_c, r_d and h_a, h_b, h_c, h_d the radii of exinscribed spheres and the altitudes in tetrahedron $ABCD$, then

1).

$$3\sum \frac{1}{h_a^4}+\frac{4}{\prod h_a}\geq \frac{1}{r}\sum \frac{1}{h_a^3}$$

2).

$$3\sum \frac{1}{r_a^4}+\frac{4}{\prod r_a}\geq \frac{2}{r}\sum \frac{1}{r_a^3}$$

where r is the radius of inscribed sphere..

Proof. In Application 3 we take:
1). $x_1=\frac{1}{h_a},x_2=\frac{1}{h_b},x_3=\frac{1}{h_c},x_4=\frac{1}{h_d}$ and $\sum \frac{1}{h_a}=\frac{1}{r}$
2). $x_1=\frac{1}{r_a},x_2=\frac{1}{r_b},x_3=\frac{1}{r_c},x_4=\frac{1}{r_d}$ and $\sum \frac{1}{r_a}=\frac{2}{r}$
The inequality of Turkevici can be generalized in following way:.

Theorem 2. *If* $x_k>0\,(k=1,2,...,n)$, *then*

$$\sum_{1\leq i<j\leq n}(x_i-x_j)^2+n\sqrt[n]{\prod_{k=1}^{n}x_k^2}\geq \sum_{k=1}^{n}x_k^2$$

Finally, we generalize the inequality of Surányi in following way:

Theorem 3. *If* $a_k\in I$ $(I\subseteq R)$ $(k=1,2,...,n)$, $f:I\to R$ *and* f *and* f' *are convex functions, then:*

$$(n-1)\sum_{k=1}^{n}f(a_k)+nf\left(\frac{1}{n}\sum_{k=1}^{n}a_k\right)\geq \sum_{i,j=1}^{n}f\left(\frac{(n-1)\,a_i+a_j}{n}\right)$$

.

Proof. We suppose that $a_1\geq a_2\geq ...\geq a_n$, so the desired inequality can be discomposed in the following two inequalities:
(1).

$$\sum_{k=1}^{n-1}(n-1-k)\,f(a_k)+\sum_{k=1}^{n-1}f\left(\frac{ka_k+a_{k+1}+...+a_n}{n}\right)\geq \sum_{1\leq i<j\leq n}f\left(\frac{(n-1)\,a_i+a_j}{n}\right)$$

and
(2).

$$\sum_{k=1}^{n-1}(k-1)\,f(a_k)+(n-2)\,f(a_n)+nf\left(\frac{a_1+a_2+...+a_n}{n}\right)$$

$$\geq \sum_{k=1}^{n-1}f\left(\frac{ka_k+a_{k+1}+...+a_n}{n}\right)+\sum_{1\leq i<j\leq n}f\left(\frac{(n-1)\,a_i+a_j}{n}\right)$$

The inequality (1) is the consequence of inequalities

$$(n-1-k)\,f(a_k)+f\left(\frac{ka_k+a_{k+1}+...+a_n}{n}\right)\geq \sum_{j=k+1}^{n}f\left(\frac{(n-1)\,a_k+a_j}{n}\right),$$

where $k\in\{1,2,...,n-1\}$ but this holds from Karamata‘s inequality using for

$$\left(a_k,a_k,...,a_k,\frac{ka_k+a_{k+1}+...+a_n}{n}\right)$$

and
$$\left(\frac{(n-1)\,a_k+a_{k+1}}{n},\frac{(n-1)\,a_k+a_{k+2}}{n},...,\frac{(n-1)\,a_k+a_n}{n}\right).$$

The inequality of Karamata says that: If $f:I\to R$ is convex $x_1\ge x_2\ge ...\ge x_n$ and $y_1\ge y_2\ge ...\ge y_n$, $x_1\ge y_1$, $x_1+x_2\ge y_1+y_2,...,x_1+x_2+...+x_{n-1}\ge y_1+y_2+...+y_{n-1}$, $x_1+x_2+...+x_n=y_1+y_2+...+y_n$, then
$$f(x_1)+f(x_2)+...+f(x_n)\ge f(y_1)+f(y_2)+...+f(y_n).$$

In our case
$$(x_1,x_2,...,x_{n-k})=\left(a_k,a_k,...,a_k,\frac{ka_k+a_{k+1}+...+a_n}{n}\right)$$
and
$$(y_1,y_2,...,y_{n-k})=\left(\frac{(n-1)\,a_k+a_{k+1}}{n},\frac{(n-1)\,a_k+a_{k+2}}{n},...,\frac{(n-1)\,a_k+a_n}{n}\right).$$

Now we prove the inequality (2).

Denote
$$F(a_1,a_2,...,a_n)=\sum_{i=1}^{n-1}(i-1)\,f(a_i)+(n-2)\,f(a_n)+nf\left(\frac{a_1+a_2+...+a_n}{n}\right)$$
$$-\sum_{i=1}^{n-1}f\left(\frac{ia_i+a_{i+1}+...+a_n}{n}\right)-\sum_{1\le i<j\le n}f\left(\frac{(n-1)\,a_i+a_j}{n}\right),$$
for which we prove that:
$$F(a_1,a_2,...,a_n)\ge F(a_2,a_2,a_3,...,a_n)\ge ...$$
$$\ge F(a_{n-1},a_{n-1},...,a_{n-1},a_n)\ge F(a_n,a_n,...,a_n)=0.$$

In $F(a_k,a_k,...,a_k,a_{k+1},a_{k+2},...,a_n)$, contain a_k the following expression
$$\sum_{i=1}^{n}(i-1)\,f(a_k)$$
$$+nf\left(\frac{ka_k+a_{k+1}+...+a_n}{n}\right)-\sum_{i=1}^{k}f\left(\frac{ka_k+a_{k+1}+...+a_n}{n}\right)$$
$$-\sum_{1\le i<j\le k}f\left(\frac{(n-1)\,a_k+a_k}{n}\right)-\sum_{j=1}^{k}\sum_{i=k+1}^{n}f\left(\frac{(n-1)\,a_i+a_k}{n}\right)$$
$$=(n-k)\,f\left(\frac{ka_k+a_{k+1}+...+a_n}{n}\right)-k\sum_{i=k+1}^{n}f\left(\frac{(n-1)\,a_i+a_k}{n}\right).$$

Denote $G_k(a)=F(a,a,...,a,a_{k+1},a_{k+2},...,a_n)$, where $a\in[a_{k+1},a_k]$, then
$$G_k'(a)=\frac{k(n-k)}{n}\left(f'\left(\frac{ka+a_{k+1}+...+a_n}{n}\right)\right.$$
$$\left.-\frac{1}{n-k}\sum_{i=k+1}^{n}f'\left(\frac{(n-1)\,a_i+a}{n}\right)\right)\ge 0,$$

because

$$\frac{ka + a_{k+1} + ... + a_n}{n} \geq \frac{1}{n-k} \sum_{i=k+1}^{n} \frac{(n-1) a_i + a}{n}$$

or

$$(n-k) a \geq \sum_{i=k+1}^{n} a_i,$$

which is true.

Sincee f is convex, then f' is increasing but f' is convex, so

$$f' \left(\frac{ka + a_{k+1} + ... + a_n}{n} \right) \geq f' \left(\frac{1}{n-k} \sum_{i=k+1}^{n} \frac{(n-1) a_i + a}{n} \right)$$

$$\geq \frac{1}{n-k} \sum_{i=k+1}^{n} f' \left(\frac{(n-1) a_i + a}{n} \right),$$

which follows from Jensen‘s inequality.

Therefore G is increasing and

$$F(a_k, a_k, ..., a_k, a_{k+1}, a_{k+2}, ..., a_n) \geq F(a_{k+1}, a_{k+1}, ..., a_{k+1}, a_{k+2}, ..., a_n)$$

which proves the affirmation..

Remark. If in Theorem 3 we take $f(a) = e^{na}$ and $e^{a_k} = x_k$ $(k = 1, 2, ..., n)$, then we obtain the inequality of Surányi..

Application 5. If $a_k > 0\,(k = 1, 2, ..., n)$ and $\alpha \geq 2$, then

$$(n-1) \sum_{k=1}^{n} a_k^{\alpha} + n \left(\frac{1}{n} \sum_{k=1}^{n} a_k \right)^{\alpha} \geq \sum_{i,j=1}^{n} \left(\frac{(n-1) a_i + a_j}{n} \right)^{\alpha}.$$

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Proof. In Theorem 3 we take $f(a) = a^{\alpha}$.

References.

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