MAJORISATION INEQUALITIES FOR STIELTJES INTEGRALS

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Abstract. Inequalities of the majorisation type for convex functions and Stieltjes integrals are given. Applications for some particular convex functions of interest are also pointed out.

1. Introduction

For fixed $n \geq 2$, let

$$\mathbf{x} = (x_1, \dots, x_n), \ \mathbf{y} = (y_1, \dots, y_n)$$

be two $n-$tuples of real numbers. Let

$$x_{[1]} \geq x_{[2]} \geq \cdots \geq x_{[n]}, \quad y_{[1]} \geq y_{[2]} \geq \cdots \geq y_{[n]},$$
$$x_{(1)} \leq x_{(2)} \leq \cdots \leq x_{(n)}, \quad y_{(1)} \leq y_{(2)} \leq \cdots \leq y_{(n)}$$

be their ordered components.

Definition 1. *The $n-$tuple $\mathbf{y}$ is said to majorise $\mathbf{x}$ (or $\mathbf{x}$ is majorised by $\mathbf{y}$, in symbols $\mathbf{y} \succ \mathbf{x}$), if*

$$\sum_{i=1}^{m} x_{[i]} \leq \sum_{i=1}^{m} y_{[i]} \quad \text{holds for } m = 1, 2, \dots, n-1; \tag{1.1}$$

and

$$\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i. \tag{1.2}$$

The following theorem is well-known in the literature as the *Majorisation Theorem,* and a convenient reference for its proof is Marshall and Olkin [6, p. 11]. This result is due to Hardy, Littlewood and Pólya [4, p. 75] and can also be found in Karamata [5]. For a discussion concerning the matter of priority see Mitrinović [7, p. 169].

Theorem 1. *Let I be an interval in $\mathbb{R}$, and let $\mathbf{x}, \mathbf{y}$ be two $n-$tuples such that $x_i, y_i \in I \ (i = 1, \dots, n)$, then*

$$\sum_{i=1}^{n} \phi(x_i) \leq \sum_{i=1}^{n} \phi(y_i) \tag{1.3}$$

holds for every continuous convex function $\phi : I \to \mathbb{R}$ iff $\mathbf{y} \succ \mathbf{x}$ holds.

The following theorem is a weighted version of Theorem 1. It can be regarded as a generalisation of the majorisation theorem and is given in Fuchs [3].

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Theorem 2. *Let* $\mathbf{x}, \mathbf{y}$ *be two decreasing* $n-$*tuples and let* $\mathbf{p} = (p_1, \dots, p_n)$ *be a real* $n-$*tuple such that*

$$\sum_{i=1}^{k} p_i x_i \leq \sum_{i=1}^{k} p_i y_i \quad \text{for} \quad k = 1, \dots, n-1, \tag{1.4}$$

and

$$\sum_{i=1}^{n} p_i x_i = \sum_{i=1}^{n} p_i y_i, \tag{1.5}$$

then for every continuous convex function $\phi : I \to \mathbb{R}$ *we have,*

$$\sum_{i=1}^{n} p_i \phi(x_i) \leq \sum_{i=1}^{n} p_i \phi(y_i). \tag{1.6}$$

Another result of this type was obtained by Bullen, Vasić and Stanković [1].

Theorem 3. *Let* $\mathbf{x}, \mathbf{y}$ *be two decreasing* $n-$*tuples and* $\mathbf{p}$ *be a real* $n-$*tuple. If*

$$\sum_{i=1}^{k} p_i x_i \leq \sum_{i=1}^{k} p_i y_i \quad \text{for} \quad k = 1, \dots, n-1, n; \tag{1.7}$$

holds, then (1.6) holds for every continuous increasing convex function $\phi : I \to \mathbb{R}$. *If* $\mathbf{x}, \mathbf{y}$ *are increasing* $n-$*tuples and the reverse inequality in (1.7) holds, then (1.6) holds for every decreasing convex function* $\phi : I \to \mathbb{R}$.

For a simple proof of Theorem 2 and Theorem 3, see [8, p. 323 – 324].

Remark 1. *It is known that (see for details* [8, p. 324]*) the conditions (1.4) and (1.5) are not necessary for (1.6) to hold. However, when the components of* $\mathbf{p}$ *are all nonnegative, then (1.4) and (1.5) (respectively (1.7)) are necessary for (1.6) to hold.*

Now, consider the continuous case. Firstly, recall some known results (see for example [8, p. 324]).

Definition 2. *Let* $x, y : [a, b] \to \mathbb{R}$ *be two given functions defined on* $[a, b]$. *The function* $y(t)$ *is said to majorise* $x(t)$, *in symbols,* $y(t) \succ x(t)$ *for* $t \in [a, b]$, *if they are decreasing in* $t \in [a, b]$ *and*

$$\int_a^s x(t)\, dt \leq \int_a^s y(t)\, dt \quad \text{for} \quad s \in [a, b] \tag{1.8}$$

and

$$\int_a^b x(t)\, dt = \int_a^b y(t)\, dt. \tag{1.9}$$

The following integral version of the Majorisation theorem holds (see for example [8, p. 325]).

Theorem 4. *The function* $x(t)$ *is majorised by* $y(t)$ *on* $[a, b]$ *if and only if they are decreasing in* $[a, b]$ *and*

$$\int_a^b \phi(x(t))\, dt \leq \int_a^b \phi(y(t))\, dt \tag{1.10}$$

holds for every ϕ *that is continuous and convex in* $[a, b]$ *such that the integrals exist.*

For other more general results due to Fan & Lorentz and Pečarić, see for example [8, p. 325-332].

The aim of this paper is to give some new majorisation type inequalities for Stieltjes integrals. Some applications are also provided.

2. Jensen Type Results for the Stieltjes Integral

Suppose that I is an interval of real numbers with interior $\mathring{I}$and $F : I \to \mathbb{R}$ is a convex function on I, then F is continuous on $\mathring{I}$ and has finite left and right derivatives at each point of $\mathring{I}$. Moreover, if $x, y \in \mathring{I}$ and $x < y$, then $D^{-}F(x) \leq D^{+}F(x) \leq D^{-}F(y) \leq D^{+}F(y)$, which shows that both $D^{-}F$ and $D^{+}F$ are nondecreasing functions on $\mathring{I}$. It is also well known that a convex function must be differentiable except for at most countably many points.

For a convex function $F : I \to \mathbb{R}$, the *subdifferential* of F denoted by ∂F is the set of all functions $\varphi : I \to [-\infty, \infty]$ such that $\varphi\left(\mathring{I}\right) \subset \mathbb{R}$ and

$$F(x) \geq F(a) + (x - a)\varphi(a) \quad \text{for any } \; x, a \in I. \tag{2.1}$$

It is also well known that if F is convex on I, then ∂F is nonempty, $D^{-}F, D^{+}F \in \partial F$ and if $\varphi \in \partial F$, then

$$D^{-}F(x) \leq \varphi(x) \leq D^{+}F(x) \tag{2.2}$$

for every $x \in \mathring{I}$. In particular, φ is a nondecreasing function.

If F is differentiable and convex on $\mathring{I}$, then $\partial F = \{F'\}$, where F' denotes the derivative of F.

The following general result holds.

Theorem 5. *Let $F : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function on the interval I and $x, y, p : [a, b] \to I$ with $p(x) \geq 0$ for $x \in [a, b]$. If $\varphi \in \partial F$, $u : [a, b] \to I$ is a monotonic nondecreasing function on $[a, b]$ and such that the following Stieltjes integrals exist*

$$\int_a^b p(t) F(x(t))\, du(t), \quad \int_a^b p(t) F(y(t))\, du(t),$$
$$\int_a^b p(t)\, x(t)\, \varphi(y(t))\, du(t), \quad \int_a^b p(t)\, y(t)\, \varphi(y(t))\, du(t),$$

then,

$$\begin{aligned}\int_a^b p(t) F(x(t))\, du(t) &- \int_a^b p(t) F(y(t))\, du(t) \\ &\geq \int_a^b p(t)\, x(t)\, \varphi(y(t))\, du(t) - \int_a^b p(t)\, y(t)\, \varphi(y(t))\, du(t).\end{aligned} \tag{2.3}$$

Proof. If we apply (2.1) for the selection $x \to x(t)$, $a \to y(t)$, we may write

$$F(x(t)) - F(y(t)) \geq (x(t) - y(t))\, \varphi(y(t)) \tag{2.4}$$

for any $t \in [a, b]$.

Multiplying (2.4) by $p(t) \geq 0$ and integrating over $u(t)$ (which is monotonically nondecreasing), we deduce the desired result (2.3). ∎

In the following we show that the above inequality possesses some particular cases that are of interest.

Corollary 1 (Jensen's Inequality)**.** *Let $F : I \subseteq \mathbb{R} \to \mathbb{R}$ be a continuous convex function on I. If $x, p : [a,b] \to I$ are continuous, $p \geq 0$ on $[a,b]$ and $u : [a,b] \to \mathbb{R}$ is monotonic nondecreasing on $[a,b]$ with $\int_a^b p(t)\,du(t) > 0$, then the Stieltjes integrals $\int_a^b p(t)\,x(t)\,du(t)$, $\int_a^b p(t)\,F(x(t))\,du(t)$ exist and*

$$\frac{1}{\int_a^b p(t)\,du(t)} \int_a^b p(t)\,F(x(t))\,du(t) \geq F\left(\frac{1}{\int_a^b p(t)\,du(t)} \int_a^b p(t)\,x(t)\,du(t)\right). \tag{2.5}$$

Proof. Follows by Theorem 5 on choosing

$$y(s) = \frac{1}{\int_a^b p(t)\,du(t)} \int_a^b p(t)\,x(t)\,du(t), \quad s \in [a,b].$$

We omit the details. ∎

The following reverse of Jensen's inequality also holds.

Corollary 2. *Let $F : I \subseteq \mathbb{R} \to \mathbb{R}$ be a continuous convex function on I, $x, p : [a,b] \to \mathbb{R}$ are continuous, $p \geq 0$ on $[a,b]$, $u : [a,b] \to \mathbb{R}$ is monotonic nondecreasing on $[a,b]$ with $\int_a^b p(t)\,du(t) > 0$. If $\varphi \in \partial F$ is such that the following Stieltjes integrals exist:*

$$\int_a^b p(t)\,y(t)\,\varphi(y(t))\,du(t) \quad \text{and} \quad \int_a^b p(t)\,\varphi(y(t))\,du(t), \tag{2.6}$$

then,

$$\begin{aligned} 0 &\leq \frac{1}{\int_a^b p(t)\,du(t)} \int_a^b p(t)\,F(y(t))\,du(t) \\ &\quad - F\left(\frac{1}{\int_a^b p(t)\,du(t)} \int_a^b p(t)\,y(t)\,du(t)\right) \\ &\leq \frac{1}{\int_a^b p(t)\,du(t)} \int_a^b p(t)\,y(t)\,\varphi(y(t))\,du(t) \\ &\quad - \frac{1}{\int_a^b p(t)\,du(t)} \int_a^b p(t)\,y(t)\,du(t) \cdot \frac{1}{\int_a^b p(t)\,du(t)} \int_a^b p(t)\,\varphi(y(t))\,du(t). \end{aligned} \tag{2.7}$$

Proof. Follows by Theorem 5 on choosing

$$x(s) := \frac{1}{\int_a^b p(t)\,du(t)} \int_a^b p(t)\,y(t)\,du(t), \quad s \in [a,b].$$

The details are omitted. ∎

Remark 2. *We observe that, the above inequality (2.7) is the corresponding version for the Stieltjes integral of the Dragomir-Ionescu reverse for the discrete Jensen's inequality obtained in 1994,* [2].

3. Majorisation Type Results

In what follows, we provide some sufficient conditions for the inequality

$$\int_a^b p(t) F(x(t))\, du(t) \geq \int_a^b p(t) F(y(t))\, du(t), \tag{3.1}$$

to hold true, where $F : I \subseteq \mathbb{R} \to \mathbb{R}$ is a convex function.

The following result may be stated:

Theorem 6. *Let $F : I \subseteq \mathbb{R} \to \mathbb{R}$ be a continuous convex function on I and $x, y, p, u : [a,b] \to I$ real functions such that*

(i) x, y, p, u are continuous on $[a,b]$;
(ii) u is monotonic nondecreasing on $[a,b]$;
(iii) p is of bounded variation on $[a,b]$.
(iv) If y is monotonic nondecreasing (nonincreasing) and $x-y$ is monotonic nondecreasing (nonincreasing) on $[a,b]$ and

$$\int_a^b p(t)\, x(t)\, du(t) = \int_a^b p(t)\, y(t)\, du(t),$$

then,

$$\int_a^b p(t) F(x(t))\, du(t) \geq \int_a^b p(t) F(y(t))\, du(t). \tag{3.2}$$

Proof. Since u is monotonic nondecreasing and the functions

$$[a,b] \ni t \longmapsto p(t) F(x(t)), \qquad [a,b] \ni t \longmapsto p(t) F(y(t))$$

are continuous on $[a,b]$, the Stieltjes integrals

$$\int_a^b p(t) F(x(t))\, du(t) \quad \text{and} \quad \int_a^b p(t) F(y(t))\, du(t)$$

exist.

Assuming that y and $x-y$ are monotonic nondecreasing, then $x = (x-y)+y$ is also monotonic nondecreasing on $[a,b]$. If $\varphi \in \partial F$, then φ is monotonic nondecreasing and thus $\varphi \circ y$ is also monotonic nondecreasing and, a fortiori, of bounded variation on $[a,b]$. Since p is of bounded variation, we deduce that the function $[a,b] \ni t \longmapsto p(t)(x(t)-y(t))\varphi(y(t))$ is of bounded variation on $[a,b]$. Further, since u is continuous, then the Stieltjes integral

$$\int_a^b p(t)(x(t)-y(t))\varphi(y(t))\, du(t)$$

exists and by (2.3),

$$\int_a^b p(t) F(x(t))\, du(t) - \int_a^b p(t) F(y(t))\, du(t) \geq \int_a^b p(t)(x(t)-y(t))\varphi(y(t))\, du(t). \tag{3.3}$$

Using the following Čebyšev type inequality

$$\frac{1}{\int_a^b p(t)\,du(t)}\int_a^b p(t)\,A(t)\,B(t)\,du(t) \geq \frac{1}{\int_a^b p(t)\,du(t)}\int_a^b p(t)\,A(t)\,du(t)\cdot\frac{1}{\int_a^b p(t)\,du(t)}\int_a^b p(t)\,B(t)\,du(t) \tag{3.4}$$

when A, B have the same monotonicity on $[a,b]$, $p \geq 0$ on $[a,b]$, p is of bounded variation on $[a,b]$, $u : [a,b] \to \mathbb{R}$ is continuous and monotonic nondecreasing on $[a,b]$ with $\int_a^b p(t)\,du(t) > 0$, we may write,

$$\frac{1}{\int_a^b p(t)\,du(t)}\int_a^b p(t)\,(x(t)-y(t))\,\varphi(y(t))\,du(t) \geq \frac{1}{\int_a^b p(t)\,du(t)}\int_a^b p(t)\,(x(t)-y(t))\,du(t) \times \frac{1}{\int_a^b p(t)\,du(t)}\int_a^b p(t)\,\varphi(y(t))\,du(t). \tag{3.5}$$

Since (iv) holds, then the right hand side of (3.5) is zero and by (3.3) and (3.5) we deduce the desired result (3.2).

The result for the case where y and $x-y$ are monotonic nonincreasing is established likewise. ∎

Remark 3. *The assumption (iv), in Theorem 6, is a strong condition for p, x, y, u. This can be relaxed if a monotonicity property for the convex function F is assumed.*

The following theorem also holds.

Theorem 7. *With the assumptions of Theorem 6 but instead of (iv) we assume that*

$$\int_a^b p(t)\,x(t)\,du(t) \geq \int_a^b p(t)\,y(t)\,du(t), \tag{iv'}$$

and $F : I \subseteq \mathbb{R} \to \mathbb{R}$ is monotonic nondecreasing on I, then (3.2) holds true.

Proof. The proof is as in Theorem 6 noting that by (iv') and by the monotonicity on F we have

$$\int_a^b p(t)\,(x(t)-y(t))\,du(t) \geq 0, \qquad \int_a^b p(t)\,\varphi(y(t))\,du(t) \geq 0,$$

implying that

$$\int_a^b p(t)\,(x(t)-y(t))\,\varphi(y(t))\,du(t) \geq 0.$$

This completes the proof. ∎

4. Applications

The following result may be stated.

Proposition 1. *Let $G : I \subseteq \mathbb{R} \to \mathbb{R}$ be a twice differentiable function on $\mathring{I}$ and such that there exist constants $\gamma, \Gamma \in \mathbb{R}$ with the property that*

$$\gamma \leq \frac{d^2 G(x)}{dt^2} \leq \Gamma \quad \text{for any} \quad x \in \mathring{I}. \tag{4.1}$$

If x, y, p, u satisfy the conditions (i) – (iv) from Theorem 6, then

$$\begin{aligned} &\frac{1}{2}\Gamma \int_a^b p(t) \left[x^2(t) - y^2(t)\right] du(t) \\ &\geq \int_a^b p(t) G(x(t)) \, du(t) - \int_a^b p(t) G(y(t)) \, du(t) \\ &\geq \frac{1}{2}\gamma \int_a^b p(t) \left[x^2(t) - y^2(t)\right] du(t). \end{aligned} \tag{4.2}$$

Proof. Consider the auxiliary function $F_\Gamma : I \to \mathbb{R}$, $F_\Gamma(x) := \frac{1}{2}\Gamma x^2 - G(x)$, $x \in I$. Since G is twice differentiable, so too is F_Γ and

$$\frac{d^2 F_\Gamma}{dt^2}(x) = \Gamma - \frac{d^2 G(x)}{dt^2} \geq 0 \quad \text{for any} \quad x \in \mathring{I},$$

showing that F_Γ is a twice differentiable and convex function on $\mathring{I}$.

Applying Theorem 6 for F_Γ we deduce the first inequality in (4.2).

The second inequality may be obtained in a similar way on making use of the function $F_\gamma : I \subseteq \mathbb{R} \to \mathbb{R}$, $F_\gamma(x) := G(x) - \frac{1}{2}\gamma x^2$, which is also twice differentiable and convex on $\mathring{I}$. ∎

Remark 4. *We observe that, if (i) – (iv) of Theorem 6 are valid, then by the Čebyšev's inequality (3.4) we have*

$$\begin{aligned} &\int_a^b p(t) \left[x^2(t) - y^2(t)\right] du(t) \\ &= \int_a^b p(t) (x(t) - y(t)) (x(t) + y(t)) \, du(t) \\ &\geq \frac{1}{\int_a^b p(t) \, du(t)} \int_a^b p(t) [x(t) - y(t)] \, du(t) \int_a^b p(t) [x(t) + y(t)] \, du(t) \\ &= 0. \end{aligned}$$

Consequently, if $\gamma > 0$ in (4.1), then by (4.2) we have

$$\begin{aligned} &\int_a^b p(t) G(x(t)) \, du(t) - \int_a^b p(t) G(y(t)) \, du(t) \\ &\qquad \geq \frac{1}{2}\gamma \int_a^b p(t) \left[x^2(t) - y^2(t)\right] du(t) \geq 0, \end{aligned} \tag{4.3}$$

which provides a refinement for the majorisation type inequality (3.2).

The following result also holds.

Proposition 2. *Let $G : I \subseteq (0,\infty) \to \mathbb{R}$ be a twice differentiable function on $\mathring{I}$ such that there exist constants δ, Δ with*

$$\delta \leq x \frac{d^2 G(x)}{dt^2} \leq \Delta \quad \textit{for any} \quad x \in \mathring{I}. \tag{4.4}$$

If x, y, p, u satisfy the conditions (i) – (iv) from Theorem 6, then

$$\begin{aligned} &\Delta \int_a^b p(t)\left[x(t)\ln x(x) - y(t)\ln y(t)\right] du(t) \\ &\geq \int_a^b p(t)\, G(x(t))\, du(t) - \int_a^b p(t)\, G(y(t))\, du(t) \\ &\geq \delta \int_a^b p(t)\left[x(t)\ln x(x) - y(t)\ln y(t)\right] du(t). \end{aligned} \tag{4.5}$$

The proof is similar to the one incorporated in Proposition 1 on utilising the auxiliary functions $F_\Delta,\ F_\delta : I \subseteq (0,\infty) \to \mathbb{R}$, $F_\Delta(x) = \Delta x \ln x - G(x)$, $F_\delta(x) = G(x) - \delta x \ln x$, which are twice differentiable and convex on $\mathring{I}$.

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