PROBLEMS OF DIFFRACTION TYPE FOR ELLIPTIC PSEUDO-DIFFERENTIAL OPERATORS WITH VARIABLE SYMBOLS

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Abstract. In this paper we consider problems of diffraction type for elliptic pseudo-differential operators with variable symbols depending on parameters. We compare the regularizators of a diffraction and a Dirichlet problem, and we prove that the regularizator of a diffraction problem tends to the regularizator of a Dirichlet problem as the parameter of the external domain tends to zero.

1. Introduction

In this paper we consider problems of diffraction type for elliptic pseudo-differential operators. In more details, we consider simultaneously two pseudo-differential equations elliptic with parameters in different domains with a common boundary. A classical diffraction problem for differential operators was considered, for example, by A.N.Tichonov and A.A. Samarsky ([6]). In the statement of this problem, the homogeneity of a medium is broken by a bounded domain provided that the solution satisfies the conditions of a maximal smoothness on the boundary of this domain. In [3] the analogous problem for pseudo-differential equations was studied, but the main result was obtained only for the case of pseudo-differential operators with constant symbols. In this article we consider the same problem for pseudo-differential equations with variable symbols depending on two parameters, under the condition that one of the parameters tends to infinity.

For example, we consider a diffraction problem in $\mathbb{R}^n_+ = \{x \in \mathbb{R}^n, x_n \geq 0\}$ and in $\mathbb{R}^n_-$ (where $\mathbb{R}^n_- = \mathbb{R} - \mathbb{R}^n_+$) as follows:

$$\left\{ \begin{array}{l} P^+A(x,D,q)u_+ = f_+, \;\; x \in \mathbb{R}^n_+ \\ P^-B(x,D,p)u_- = f_-, \;\; x \in \mathbb{R}^n_- \end{array} \right. \tag{1.1}$$

where A and B are pseudo-differential operators of order m_1 and m_2 elliptic with parameter q and p, respectively. If p is big, then the solution in the half space $\mathbb{R}^n_-$ has the form of a boundary layer with respect to x_n. For instance, the function $e^{\frac{x_n}{\varepsilon}}$ $(x_n < 0)$ is boundary layer function. If $\varepsilon = 1/p$ tends to zero, this function approaches zero for $x_n < 0$.

It is possible to prove that if the symbols of operators A and B don't depend on x, then we can find an exact solution of problem (1.1) (see [3])which is defined by the inverse operator. That is, if we write the problem (1.1) in the form

$$\mathfrak{A}u = \mathfrak{f}$$

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where

$$\mathfrak{A} = \left\{ P^+ A, P^- B \right\},$$

then

$$u = \mathfrak{A}^{-1} \mathfrak{f}$$

In the case when A and B depend on x, the inverse operator can not be defined explicitly but if we can find an operator $\mathfrak{R}$ such that

$$\mathfrak{R}\mathfrak{f} = \mathfrak{f} + \mathfrak{T}\mathfrak{f}$$

where the operator $\mathfrak{T}$ has the small norm, we say that the operator $\mathfrak{R}$ is the regularizator [1] of problem (1.1).

We are going to evaluate the difference between the regularizators for problem (1.1) and the Dirichlet problem (1.2) below:

$$P^+ A(D, x, q) u_+^{(0)} = f_+(x) \tag{1.2}$$

We prove that the regularizator of the Dirichlet problem (1.2) can be obtained as a limit case in the diffraction problem (1.1) as $p = (1/\varepsilon)$ tends to infinity ($\varepsilon \to 0$). We shall use the technique of the theory of pseudo-differential operators developed in [5], [7] and the notations of [2].

2. Notations and properties

Let $\mathcal{H}_{l_1,l_2}(\mathbb{R}^n)$ be a space of distributions $u(x)$,

$$x = (x', x_n) = (x_1, x_2, ..., x_{n-1}, x_n) \in \mathbb{R}^n$$

with the norm

$$\|u(x)\|_{l_1,l_2} = \left\| (q + |\xi|)^{l_1} (p + |\xi|)^{l_2} \tilde{u}(\xi) \right\|_{\mathcal{L}_2} \tag{2.1}$$

where p, q are real non-negative parameters,

$$\xi = (\xi', \xi_n) = (\xi_1, \xi_2, ...\xi_n), \ \langle x, \xi \rangle = x_1\xi_1 + x_2\xi_2 + \cdots\cdots x_n\xi_n;$$

$$|\xi| = \sqrt{|\xi'|^2 + \xi_n^2} = \sqrt{\xi_1^2 + \xi_2^2 + \cdots + \xi_{n-1}^2 + \xi_n^2}$$

and

$$\tilde{u}(\xi) = \mathcal{F}_{x \to \xi}[u(x)] = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{i\langle x, \xi \rangle} u(x) dx \tag{2.2}$$

The norm on the right-hand side of (2.1) is the usual norm in $\mathcal{L}_2\left(\mathbb{R}^n_\xi\right)$.

If $p = q = 1$, then the space $\mathcal{H}_{l_1,l_2}(\mathbb{R}^n)$ coincides with the ordinary Sobolev space $\mathcal{H}_{l_1+l_2}(\mathbb{R}^n) = \mathcal{W}^{(2)}_{l_1+l_2}(\mathbb{R}^n)$. Since $\mathcal{L}_2$ and $\mathcal{H}_0$ are the notations of the same space we shall write further $\|\cdot\|_0$ instead of $\|\cdot\|_{\mathcal{L}_2}$. We introduce also the spaces $\mathcal{H}_s(\mathbb{R}^n_+)$ and $\mathcal{H}_s(\mathbb{R}^n_-)$ of functions f_+ and f_- defined in $\mathbb{R}^n_+ = \{x \in \mathbb{R}^n : x_n > 0\}, \mathbb{R}^n_- = \{x \in \mathbb{R}^n : x_n < 0\}$, $\mathbb{R}^n_- = \mathbb{R}^n \backslash \mathbb{R}^n_+$, respectively, with the norms

$$\|f_+\|_s^+ = \left\| \Pi^+ \left(\xi_n - i\left|\xi'_q\right|\right)^s \widetilde{Ef_+} \right\|_0, \ \|f_-\|_s^- = \left\| \Pi^- \left(\xi_n - i\left|\xi'_p\right|\right)^s \widetilde{Ef_-} \right\|_0$$

where

$$\left|\xi'_q\right| = \sqrt{|\xi'|^2 + q^2}, \quad \left|\xi'_p\right| = \sqrt{|\xi'|^2 + p^2},$$

[1] More general definition of regularizator is given, for example, in [3] or [7].

$$\Pi^{\pm}\tilde{u}(\xi) = \mathcal{F}_{x\to\xi}\left[\theta^{\pm}u(x)\right] = \pm\frac{i}{2\pi}\int_{-\infty}^{\infty}\frac{\tilde{u}(\xi',\eta_n)}{\xi_n+i0-\eta_n}d\eta_n$$
$$= \pm\frac{i}{2\pi}V.P.\int_{-\infty}^{\infty}\frac{\tilde{u}(\xi',\eta_n)}{\xi_n+i0-\eta_n}d\eta_n+\frac{1}{2}\tilde{u}(0) \qquad (2.3)$$

and

$$\theta^{+}(x) = \begin{cases} 1, & \text{if } x_n \geq 0 \\ 0, & \text{if } x_n < 0 \end{cases}, \ \theta^{-}(x) = \begin{cases} 0, & \text{if } x_n > 0 \\ 1, & \text{if } x_n < 0 \end{cases}$$

and $\widetilde{Ef}_{\pm}$ is the extension of the function $f_{\pm}$ on the whole Euclidean space $\mathbb{R}^n$ such that the extension belongs to $\mathcal{H}_{l_1,l_2}(\mathbb{R}^n)$.

We state some properties of the operator $\Pi^{\pm}$:

- (1) The operator $\Pi^{\pm}$ is defined on smooth decreasing functions by the formula (2.3). Since the operator of multiplication of the Heaviside function $\theta^{\pm}(x)$ is bounded in $\mathcal{H}_0(\mathbb{R}^n_x)$, the operator $\Pi^{\pm}$ is bounded in the space $\mathcal{H}_0\left(\mathbb{R}^n_\xi\right)$ being the dual of $\mathcal{H}_0(\mathbb{R}^n_x)$ with respect to the Fourier transform. For arbitrary function $\tilde{u}(\xi)\in\mathcal{H}_0\left(\mathbb{R}^n_\xi\right)$ the formula (2.3) is understood as the closure of the opeator $\Pi^{\pm}$.
  (2) If $\tilde{u}(\xi)\in\mathcal{H}_0\left(\mathbb{R}^n_\xi\right)$, then this function can be represented as the sum $\tilde{u}(\xi)=\tilde{u}_+(\xi)+\tilde{u}_-(\xi)$, where $\tilde{u}_{\pm}(\xi)=\Pi^{\pm}\tilde{u}(\xi)$.
  (3) Since $\theta^{+}(x)=0$ for $x_n<0$ $\left(\theta^{-}(x)=0 \text{ for } x_n>0\right)$, the function $\Pi^{+}\tilde{u}(\xi)$ $(\Pi^{-}\tilde{u}(\xi))$ admits an analytic continuation in the half-plane $\mathrm{Im}\xi_n>0$ $(\mathrm{Im}\xi_n<0)$.
  (4) If a function $\tilde{v}_+(\xi)$ $(\tilde{v}_-(\xi))\in\mathcal{H}_0\left(\mathbb{R}^n_\xi\right)$ and may be extended in the half-plane $\mathrm{Im}\xi_n>0$ $(\mathrm{Im}\xi_n<0)$, then $\Pi^{\pm}\tilde{v}_{\mp}=0$
  (5) If the functions $\Pi^{\pm}\tilde{u}(\xi)$ and $\Pi^{\pm}\left[\tilde{v}_{\pm}(\xi)\tilde{u}(\xi)\right]$ make sense, where $\tilde{v}_+(\xi)$ $(\tilde{v}_-(\xi))$ admit an analytic continuation in the half-plane $\mathrm{Im}\xi_n>0$ $(\mathrm{Im}\xi_n<0)$), then

$$\Pi^{\pm}\left[\tilde{v}_{\pm}(\xi)\tilde{u}(\xi)\right]=\Pi^{\pm}\left[\tilde{v}_{\pm}(\xi)\Pi^{\pm}\tilde{u}(\xi)\right]$$

Let $\mathfrak{f}=\{f_+,f_-\}\in\mathcal{H}_{l_1}\left(\mathbb{R}^n_+\right)\times\mathcal{H}_{l_2}\left(\mathbb{R}^n_-\right)$. On this product space we can introduce a natural operation of addition and multiplication by a function $\varphi\in\mathcal{C}^\infty(\mathbb{R}^n)$ by the following rule: If $\mathfrak{f}=\{f_+,f_-\}$ and $\mathfrak{g}=\{g_+,g_-\}$, then $\mathfrak{f}+\mathfrak{g}=\{f_++g_+,f_-+g_-\}$ and $\varphi\mathfrak{f}=\{\varphi f_+,\varphi f_-\}$. We can also introduce a natural norm on this set.

Let A and B be two pseudo-differential operators whose symbols are $\sigma(A)=a(x,\xi,q)$ and $\sigma(B)=b(x,\xi,p)$, respectively. Recall that a pseudo-differential operator corresponding to the symbol $a(x,\xi)$ is defined by

$$A(x,D)u\equiv(Au)(x)=(2\pi)^{-\frac{n}{2}}\int_{\mathbb{R}^n}e^{-i\langle x,\xi\rangle}a(x,\xi)\tilde{u}(\xi)\,d\xi \qquad (2.4)$$

We suppose that the symbols a and b depend on parameters q and p (where $q\leq p$), respectively, and satisfy the following conditions:

- (1) $a(x,\xi,q)\in\mathcal{C}^\infty\left[\mathbb{R}^n_x\times\left(\mathbb{R}^{n+1}_{\xi,q}\backslash 0\right)\right]$, $b(x,\xi,p)\in\mathcal{C}^\infty\left[\mathbb{R}^n_x\times\left(\mathbb{R}^{n+1}_{\xi,p}\backslash 0\right)\right]$.
  (2) The functions $a(x,\xi,q)$ and $b(x,\xi,p)$ are homogeneous of order m_1 and m_2, (m_1 and m_2 are positive) with respect to ξ,q and ξ,p, respectively.

(3) The operators A and B are elliptic with parameter, i.e. $a(x,\xi,q)\neq 0$ for real ξ and for $q+|\xi|\neq 0$, and $b(x,\xi,p)\neq 0$ for real ξ and for $p+|\xi|\neq 0$.

(4) For every value of multi-indexes $\alpha=(\alpha_1,\alpha_2,...,\alpha_n)\,,\beta=(\beta_1,\beta_2,...,\beta_n)\,,$ the following estimations hold:

$$\begin{aligned} \left|\partial^\alpha D^\beta a(x,\xi,q)\right| &< C^1_{\alpha,\beta}\,(q+|\xi|)^{m_1-|\alpha|}\,, \\ \left|\partial^\alpha D^\beta b(x,\xi,p)\right| &< C^2_{\alpha,\beta}\,(p+|\xi|)^{m_2-|\alpha|} \end{aligned} \tag{2.5}$$

where

$$\begin{aligned} \partial^\alpha &= \left[\frac{\partial}{\partial\xi_1}\right]^{\alpha_1}\left[\frac{\partial}{\partial\xi_2}\right]^{\alpha_2}\cdots\left[\frac{\partial}{\partial\xi_n}\right]^{\alpha_n}, \\ D^\beta &= \left[-i\frac{\partial}{\partial x_1}\right]^{\beta_1}\left[-i\frac{\partial}{\partial x_2}\right]^{\beta_2}\cdots\left[-i\frac{\partial}{\partial x_n}\right]^{\beta_n} \end{aligned}$$

and

$$|\alpha|=\alpha_1+\alpha_2+\cdots+\alpha_n,\;\; |\beta|=\beta_1+\beta_2+\cdots+\beta_n.$$

(5) The symbols a and b can be represented in the form

$$\begin{aligned} a(x,\xi,q) &= a(\infty,\xi,q)+a'(x,\xi,q), \\ b(x,\xi,p) &= b(\infty,\xi,p)+b'(x,\xi,p) \end{aligned}$$

where $a'(x,\xi,q)$ and $b'(x,\xi,p)$ are infinitely differentiable functions with respect to x, with compact support, i.e. they belong to $\mathcal{C}_0^\infty\left(\mathbb{R}^n_x\right).$

We remark that a pseudo-differential operator with a symbol satisfying the condition (5) can be defined by the following formula, which is equivalent to the formula (2.4):

$$\left(\tilde{A}u\right)(\xi)=a(\infty,\xi,q)\tilde{u}\,(\xi)+\int_{\mathbb{R}^n}\widetilde{a'}(\xi-\eta,\eta,q)\tilde{u}\,(\eta)\,d\eta \tag{2.5}$$

where the tilde "~" dentoes the Fourier transform with respect to the first argument. In [7] M.Vishik and G. Eskin have proved that symbols satisfying the conditions $(1)-(5)$ admit the following factorization:

$$a(x,\xi,q)=a_+(x,\xi',\xi_n,q)a_-(x,\xi',\xi_n,q), \tag{2.6}$$

and

$$b(x,\xi,p)=b_+(x,\xi',\xi_n,p)b_-(x,\xi',\xi_n,p) \tag{2.7}$$

where $a_+(x,\xi',\xi_n,q)$, $b_+(x,\xi',\xi_n,p)$ $\left(a_-(x,\xi',\xi_n,q),\; b_-(x,\xi',\xi_n,p)\right)$ are functions admitting an analytic continuation in the half-plane $\mathrm{Im}\xi_n>0$ ($\mathrm{Im}\xi_n<0$) and they remain homogeneous with respect to ξ,q (ξ,p). Suppose that

$$\text{ord } a_+(x,\xi',\xi_n,q)=\kappa_1,\;\text{ord } b_-(x,\xi',\xi_n,p)=\kappa_2\geq 0,\;\;(\kappa=\kappa_1+\kappa_2>0)$$

and the orders do not depend on x.

3. Evaluation of the difference between the regularizators of diffraction and of Dirichlet Problems

Consider a function

$$\mathfrak{f}\in\mathfrak{H}_{\kappa-m}=\mathcal{H}_{\kappa-m_1}\left(\mathbb{R}^n_+\right)\times\mathcal{H}_{\kappa-m_2}\left(\mathbb{R}^n_-\right)$$

with the norm

$$|\mathfrak{f}|_{\kappa-m}=\left\|\Pi^+\mu_-\left(\xi',\xi_n,q,p\right)\widetilde{Ef_+}\right\|_0+\left\|\Pi^-\mu_+\left(\xi',\xi_n,q,p\right)\widetilde{Ef_-}\right\|_0$$

where

$$\begin{aligned}\mu_-\left(\xi',\xi_n,q,p\right) &= \left(\xi_n-i\left|\xi'_q\right|\right)^{\kappa_1-m_1}\left(\xi_n-i\left|\xi'_p\right|\right)^{\kappa_2},\\ \mu_+\left(\xi',\xi_n,q,p\right) &= \left(\xi_n+i\left|\xi'_q\right|\right)^{\kappa_1}\left(\xi_n+i\left|\xi'_p\right|\right)^{\kappa_2-m_2}.\end{aligned}$$

We also introduce the couple operator ([3])

$$\mathfrak{A}u=\{P^+Au,P^-Bu\}$$

where P^+ (P^-) is the restriction operator of distributions on $\mathbb{R}_+$ $(\mathbb{R}_-)$ (it is clear that for ordinary functions it coincides with Heaviside function θ^+ $\left(\theta^-\right)$) and the operator A (B) has the symbol $a(x,\xi,q)$ $(b(x,\xi,p))$.

We consider the following diffraction problem

$$\mathfrak{A}u=\mathfrak{f}\in\mathfrak{H}_{\kappa-m},\ u\in\mathcal{H}_\kappa\left(\mathbb{R}^n\right) \tag{3.1}$$

It follows from [3] that problem (3.1) has a unique solution for sufficiently large values of parameters p and q. The proof is based on construction of the regularizator of this problem which has the following form:

$$\mathfrak{R}\mathfrak{f}=\mathcal{R}_1\left[\theta^+\mathcal{R}_-Ef_++\theta^-\mathcal{R}_+Ef_-\right]$$

Or equivalently,

$$\mathfrak{R}\mathfrak{f}=\frac{1}{A_+(x,D,q)B_-(x,D,p)}\left[\theta^+\frac{B_-(x,D,p)}{A_-(x,D,q)}Ef_+(x)+\theta^-\frac{A_+(x,D,q)}{B_+(x,D,p)}Ef_-(x)\right] \tag{3.2}$$

where $\mathcal{R}_1$, $\mathcal{R}_-$ and $\mathcal{R}_+$ are pseudo-differential operators with symbols

$$\left[a_+(x,\xi',\xi_n,q)b_-(x,\xi',\xi_n,p)\right]^{-1},b_-(x,\xi',\xi_n,p)\left[a_-(x,\xi',\xi_n,q)\right]^{-1}$$

and

$$a_+(x,\xi',\xi_n,q)\left[b_+(x,\xi',\xi_n,p)\right]^{-1}$$

respectively.

Consider at the same time with problem (3.1) the following Dirichlet problem

$$P^+A(D,x,q)u_+^{(0)}=f_+(x),\quad u_+^{(0)}\in\mathcal{H}_{\kappa_1}\left(\mathbb{R}^n_+\right) \tag{3.3}$$

here $\mathcal{H}_{\kappa_1}\left(\mathbb{R}^n_+\right)$ is the subspace of the space $\mathcal{H}_{\kappa_1}\left(\mathbb{R}^n\right)$ $(\kappa_1\geq 0)$ of functions, which vanish on $\mathbb{R}^n_-$. The regularizator of this equation was constructed by M.Vishik and G. Eskin in [7] and it has the form:

$$R_+f_+=\frac{1}{A_+(x,D,q)}\theta^+\frac{1}{A_-(x,D,q)}Ef_+(x) \tag{3.4}$$

We shall prove that if $p\to\infty$,then $\mathfrak{R}\mathfrak{f}\to R_+f_+$. It means that the regularizator of Dirichlet problem (3.3) may be obtained as a limit case of the problem (3.1) when

p approaches infinity. We represent the difference of these two operators (3.2) and (3.4) as follows:

$$\begin{aligned} \mathcal{I}f &\equiv \Re\mathfrak{f} - R_+ f_+ \qquad (3.5)\\ &= \frac{1}{A_+(x,D,q)}\left[\frac{1}{B_-(x,D,p)}\theta^+\frac{B_-(x,D,p)}{A_-(x,D,q)}Ef_+(x) - \theta^+\frac{1}{A_-(x,D,q)}Ef_+(x)\right]\\ &\quad + \frac{1}{A_+(x,D,q)B_-(x,D,p)}\theta^-\frac{A_+(x,D,q)}{B_+(x,D,p)}Ef_-(x) \end{aligned}$$

Since the smoothness of this difference is κ_1, we estimate the norm of $\mathcal{I}f$ in the space $\mathcal{H}_{\kappa_1}(\mathbb{R}^n)$. We have

$$\begin{aligned} &\left\|\Re\mathfrak{f} - R_+ f_+\right\|_{\kappa_1}\\ =\ &\left\|\left(\xi_n - i\left|\xi_q'\right|\right)^{\kappa_1}\left(\Re\mathfrak{f} - \Re_+\mathfrak{f}_+\right)\right\|_0\\ \leq\ & C\left\|\frac{1}{B_-(x,D,p)}\theta^+\frac{B_-(x,D,p)}{A_-(x,D,q)}Ef_+(x) - \theta^+\frac{1}{A_-(x,D,q)}Ef_+(x)\right\|_0 \qquad (3.6)\\ &+ C\left\|\frac{1}{B_-(x,D,p)}\theta^-\frac{A_+(x,D,q)}{B_+(x,D,p)}Ef_-(x)\right\|_0 \end{aligned}$$

Using $1 = \theta^+ + \theta^-$, we transform the term $\frac{1}{B_-(x,D,p)}\theta^+\frac{B_-(x,D,p)}{A_-(x,D,q)}Ef_+(x)$ as follows

$$\begin{aligned} &\frac{1}{B_-(x,D,p)}\theta^+\frac{B_-(x,D,p)}{A_-(x,D,q)}Ef_+(x)\\ =\ &\theta^+\frac{1}{A_-(x,D,q)}Ef_+(x) + \theta^-\frac{1}{B_-(x,D,p)}\theta^+\frac{B_-(x,D,p)}{A_-(x,D,q)}Ef_+(x) \qquad (3.7) \end{aligned}$$

Substituting (3.7) into (3.6) we obtain

$$\left\|\Re\mathfrak{f} - R_+ f_+\right\|_{\kappa_1} \leq CN_1 + CN_2 \tag{3.8}$$

where

$$\begin{aligned} N_1 &= \left\|\theta^-\frac{1}{B_-(x,D,p)}\theta^+\frac{B_-(x,D,p)}{A_-(x,D,q)}Ef_+(x)\right\|_0\\ N_2 &= \left\|\frac{1}{B_-(x,D,p)}\theta^-\frac{A_+(x,D,q)}{B_+(x,D,p)}Ef_-(x)\right\|_0 \qquad (3.9) \end{aligned}$$

We consider separately the operator $\frac{1}{B_-(x,D,p)}$. Let us set $p = 1/\varepsilon$ and transform this operator as follows:

(3.10)
$$\frac{1}{B_-(x,D,p)} = \frac{\varepsilon^{\kappa_2}}{B_-(x,\varepsilon D,1)} = \varepsilon^{\kappa_2}\frac{1}{B_-(x,0,1)}\left[1 - \frac{B_-(x,\varepsilon D,1) - B_-(x,0,1)}{B_-(x,\varepsilon D,1)}\right]$$

Moreover we have

$$\left\|\frac{1}{B_-(x,D,p)}\right\| \leq C\frac{1}{\left(p + \left|\xi'\right|\right)^{\kappa_2}} = C\varepsilon^{\kappa_2}\frac{1}{\left(1 + \varepsilon\left|\xi'\right|\right)^{\kappa_2}} \leq C\varepsilon^{\kappa_2}$$

Consequently, for N_2 we have

$$N_2 \leq C\varepsilon^{\kappa_2}\left\|\Pi^-\left(\xi_n + i\left|\xi_q'\right|\right)^{\kappa_1}\left(\xi_n + i\left|\xi_p'\right|\right)^{\kappa_2 - m_2}\widetilde{Ef}_-\right\|_0 \tag{3.11}$$

Substituting (3.10) into (3.9), for N_1 we obtain

$$\begin{aligned} N_1 &= \varepsilon^{\kappa_2} \left\| \theta^- \frac{1}{B_-(x,0,1)} \theta^+ \frac{B_-(x,D,p)}{A_-(x,D,q)} Ef_+(x) - \theta^- T_- \theta^+ \frac{B_-(x,D,p)}{A_-(x,D,q)} Ef_+(x) \right\|_0 \\ &= \left\| \theta^- T_- \theta^+ \frac{B_-(x,\varepsilon D,1)}{A_-(x,D,q)} Ef_+(x) \right\|_0 \end{aligned} \tag{3.12}$$

where we denote

$$T_- = \frac{B_-(x,\varepsilon D,1) - B_-(x,0,1)}{B_-(x,0,1) B_-(x,\varepsilon D,1)} \tag{3.13}$$

with the symbol

$$\sigma(T_-) = \frac{b_-(x,\varepsilon\xi,1) - b_-(x,0,1)}{b_-(x,0,1) b_-(x,\varepsilon\xi,1)} \tag{3.14}$$

We expand this symbol $\sigma(T_-)$ as follows

$$\sigma(T_-) = \frac{\varepsilon \sum_{k=1}^{n} \partial_k b_-(x,\varepsilon\theta\xi,1)\xi_k}{b_-(x,0,1) b_-(x,\varepsilon\xi,1)} \tag{3.15}$$

By virtue of assumption (4) for homogeneous symbols given in section 2, we have the following inequality

$$\begin{aligned} |\sigma(T_-)| &\leq C\varepsilon \frac{(1+\varepsilon|\xi|)^{\kappa_2 - 1}|\xi|}{(1+\varepsilon|\xi|)^{\kappa_2}} = C\varepsilon \frac{|\xi|}{1+\varepsilon|\xi|} \\ &\leq C\varepsilon \frac{|\xi_n|}{1+\varepsilon|\xi_n|} + C\varepsilon |\xi'| \\ &\leq C \left| \frac{-i\varepsilon}{(\varepsilon\xi_n - i)} (-i\xi_n) \right| + C\varepsilon|\xi'| \end{aligned} \tag{3.16}$$

Denoting

$$\tilde{h}(\xi,\varepsilon) = \mathcal{F}\left[\frac{B_-(x,\varepsilon D,1)}{A_-(x,D,q)} Ef_+(x) \right] = \mathcal{F}\left[h(x,\varepsilon) \right] \tag{3.17}$$

and applying the estimation (3.16) to (3.12), by the extension theory we can obtain

$$N_1 \leq C \left\| \Pi^- \sigma(T_-) \Pi^+ \tilde{h}(\xi,\varepsilon) \right\|_0 \leq C \left\| \sigma(T_-) \Pi^+ \tilde{h}(\xi,\varepsilon) \right\|_0 \leq C N_3 + C\varepsilon N_4 \tag{3.18}$$

where

$$N_3 = \left\| \frac{-i\varepsilon}{(\varepsilon\xi_n - i)} (-i\xi_n) \Pi^+ \tilde{h}(\xi,\varepsilon) \right\|_0, \quad N_4 = \left\| \Pi^+ |\xi'| \tilde{h}(\xi,\varepsilon) \right\|_0 \tag{3.19}$$

Considering (3.10) and (3.17) it is easy to verify that the norm N_4 admits the estimation

$$N_4 \leq C\varepsilon^{\kappa_2} \left\| \Pi^+ \left(\xi_n - i|\xi'_p|\right)^{\kappa_2} \left(\xi_n - i|\xi'_q|\right)^{\kappa_1 - m_1 + 1} \widetilde{Ef}_+ \right\|_0 \tag{3.20}$$

So it remains to evaluate N_3. We remark that

$$\mathcal{F}^{-1}\left[\frac{-i\varepsilon}{(\varepsilon\xi_n - i)} \right] = \theta^- e^{\frac{x_n}{\varepsilon}}$$

is the so called function in the type of boundary layer. It follows (3.19) that

$$\begin{aligned} N_3 &\leq C\left\|\theta^- e^{\frac{x_n}{\varepsilon}} * \frac{\partial}{\partial x_n}\theta^+ h(x,\varepsilon)\right\|_0 \\ &= C\left\|\theta^- e^{\frac{x_n}{\varepsilon}} * \left[\delta(x_n)h(x,\varepsilon)|_{x_n=0+0} + \theta^+\frac{\partial}{\partial x_n}h(x,\varepsilon)\right]\right\|_0 \end{aligned} \tag{3.21}$$

It follows that

$$N_3 \leq C\left\|\theta^- e^{\frac{x_n}{\varepsilon}}\right\|_0 \|h(x',0,\varepsilon)\|_0' + C\left\|\frac{-i\varepsilon}{(\varepsilon\xi_n - i)}\Pi^+\xi_n\tilde{h}(\xi,\varepsilon)\right\|_0 \tag{3.22}$$

Here "prime" denotes the norm over the boundary. Using the formula

$$\|h(x',0,\varepsilon)\|_0' \leq c\,\|h(x,\varepsilon)\|_{\delta+\frac{1}{2}}^+$$

where $0 < \delta < \frac{1}{2}$, "+" denotes the norm over the upper half-space. Taking into account the norm of boundary layer function

$$\left\|\theta^- e^{\frac{x_n}{\varepsilon}}\right\|_0 = \sqrt{\frac{\varepsilon}{2}}$$

it follows (3.16) that

$$N_3 \leq C\sqrt{\varepsilon}\,\|h(x,\varepsilon)\|_{\delta+\frac{1}{2}}^+ + C\varepsilon\left\|\Pi^+\left(\xi_n - i\left|\xi_q'\right|\right)\tilde{h}(\xi,\varepsilon)\right\|_0 \tag{3.23}$$

Substituting (3.17) into (3.23) we can obtain

$$\begin{aligned} N_3 \quad \leq \quad & c\varepsilon^{\kappa_2+\frac{1}{2}}\left\|\Pi^+\left(\xi_n - i\left|\xi_q'\right|\right)^{\delta+\frac{1}{2}}\left(\xi_n - i\left|\xi_p'\right|\right)^{\kappa_2}\left[\xi_n - i\left|\xi_q'\right|\right]^{\kappa_1-m_1}\widetilde{Ef}_+\right\|_0 \\ & + c\varepsilon^{\kappa_2+1}\left\|\Pi^+\left(\xi_n - i\left|\xi_p'\right|\right)^{\kappa_2}\left[\xi_n - i\left|\xi_q'\right|\right]^{\kappa_1-m_1+1}\widetilde{Ef}_+\right\|_0, \end{aligned}$$

it follows that

$$N_3 \leq C\varepsilon^{\kappa_2+\frac{1}{2}}\left\|\Pi^+\left(\xi_n - i\left|\xi_p'\right|\right)^{\kappa_2}\left(\xi_n - i\left|\xi_q'\right|\right)^{\kappa_1-m_1+1}\widetilde{Ef}_+\right\|_0 \tag{3.24}$$

Using the evaluation (3.18), (3.20) and (3.24) we obtain

$$\begin{aligned} N_1 \quad \leq \quad & C\varepsilon^{\kappa_2+\frac{1}{2}}\left\|\Pi^+\left(\xi_n - i\left|\xi_p'\right|\right)^{\kappa_2}\left(\xi_n - i\left|\xi_q'\right|\right)^{\kappa_1-m_1+1}\widetilde{Ef}_+\right\|_0 \\ & + C\varepsilon^{\kappa_2+1}\left\|\Pi^+\left(\xi_n - i\left|\xi_p'\right|\right)^{\kappa_2}\left(\xi_n - i\left|\xi_q'\right|\right)^{\kappa_1-m_1+1}\widetilde{Ef}_+\right\|_0, \end{aligned}$$

or more roughly

$$N_1 \leq C\varepsilon^{\kappa_2+\frac{1}{2}}\left\|\Pi^+\left(\xi_n - i\left|\xi_p'\right|\right)^{\kappa_2}\left(\xi_n - i\left|\xi_q'\right|\right)^{\kappa_1-m_1+1}\widetilde{Ef}_+\right\|_0 \tag{3.25}$$

Considering the inequality (3.11) for N_2 and the inequality (3.25) for N_1, it follows (3.8) that

$$\begin{aligned} & \|\Re\mathfrak{f} - R_+ f_+\|_{\kappa_1} \leq CN_1 + CN_2 \\ \leq \quad & C\varepsilon^{\kappa_2+\frac{1}{2}}\left\|\Pi^+\left(\xi_n - i\left|\xi_q'\right|\right)^{\kappa_1-m_1+1}\left(\xi_n - i\left|\xi_p'\right|\right)^{\kappa_2}\widetilde{Ef}_+\right\|_0 \\ & + C\varepsilon^{\kappa_2}\left\|\Pi^-\left(\xi_n + i\left|\xi_q'\right|\right)^{\kappa_1}\left(\xi_n + i\left|\xi_p'\right|\right)^{\kappa_2-m_2}\widetilde{Ef}_-\right\|_0 \end{aligned}$$

That is to say

$$\|\mathcal{I}\mathfrak{f}\| = \|\Re\mathfrak{f} - R_+ f_+\|_{\kappa_1} \leq C\left[\varepsilon^{\kappa_2+\frac{1}{2}}\|f_+\|_{\kappa_1-m_1+1,\kappa_2}^+ + \varepsilon^{\kappa_2}\|f_-\|_{\kappa_1,\kappa_2-m_2}^-\right] \tag{3.26}$$

Thus the following theorem is true, which is the generalization of the result in [3]:

Theorem 1. *Let*

$$\mathfrak{f} \in \{f_+, f_-\} \in \mathcal{H}_{\kappa_1 - m_1 + 1, \kappa_2}\left(\mathbb{R}^n_+\right) \times \mathcal{H}_{\kappa_1, \kappa_2 - m_2}\left(\mathbb{R}^n_-\right) \equiv \mathcal{H} \tag{3.27}$$

and $\Re$ *be the regularizator of problem (3.1) provided the condition* $f \in \mathfrak{H}_{\kappa - m}$ *is replaced by (3.27). Further, let* R_+ *be the regularizator of problem (3.3) with* $f_+ \in \mathcal{H}_{\kappa_1 - m_1 + 1, \kappa_2}\left(\mathbb{R}^n_+\right)$*, then for the operator* $\mathcal{I}$

$$\mathcal{I}\mathfrak{f} = \Re\mathfrak{f} - R_+ f_+$$

defined by (3.5), the estimation (3.26) is true.

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