COUNTING PRIMES IN THE INTERVAL $\left(n^2, (n+1)^2\right)$

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Abstract. In this note, we show that there are many infinity positive integer values of n, in which the following inequality holds

$$\left\lfloor \frac{1}{2}\left(\frac{(n+1)^2}{\ln(n+1)} - \frac{n^2}{\ln n}\right) - \frac{\ln^2 n}{\ln\ln n} \right\rfloor \leq \pi\big((n+1)^2\big) - \pi(n^2).$$

1. Introduction

Considering Euclid's proof for the existence many infinity primes, we can get the following inequality for many infinite values of n:

$$1 \leq \pi\big((n+1)^2\big) - \pi(n^2),$$

in which $\pi(x) = \#[2, x] \cap \mathbb{P}$, and $\mathbb{P}$ is set of all primes. Now, we have some strong results, which allow us to change 1 in left hand side of above inequality by a nontrivial one. In fact, we show that there are many infinity positive integer values of n, in which the following inequality holds:

$$\left\lfloor \frac{1}{2}\left(\frac{(n+1)^2}{\ln(n+1)} - \frac{n^2}{\ln n}\right) - \frac{\ln^2 n}{\ln\ln n} \right\rfloor \leq \pi\big((n+1)^2\big) - \pi(n^2).$$

This is the result of an unsuccessful challenge, for proving the old-famous conjecture, which asserts for every $n \in \mathbb{N}$, the interval $\left(n^2, (n+1)^2\right)$ contains at least a prime. Surely, Prime Number Theorem [1], suggests a few more number of primes as follows:

$$F(n) \sim \frac{1}{2}\left(\frac{(n+1)^2}{\ln(n+1)} - \frac{n^2}{\ln n}\right) \qquad (n \to \infty),$$

in which $F(n)$ is the number of primes in $\left(n^2, (n+1)^2\right)$. This asymptotic relation, led us to make some conjectures on the bounding $F(n)$.

Conjecture 1. For every $n \geq 5$, we have

$$F(n) < \frac{1}{2}\left(\frac{(n+1)^2}{\ln(n+1)} - \frac{n^2}{\ln n}\right) + \ln^2 n \ln\ln n.$$

This conjecture has been checked by Maple for all $5 \leq n \leq 10000$.

Conjecture 2. For every $n \geq 3$, we have

$$\frac{1}{2}\left(\frac{(n+1)^2}{\ln(n+1)} - \frac{n^2}{\ln n}\right) - \frac{\ln^2 n}{\ln\ln n} - 1 < F(n).$$

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This conjecture has been checked by Maple for all $3 \le n \le 10000$. Also, as mentioned above, we show that for many infinity positive integer values of n, the truth of this conjecture holds. To do this, we need the following sharp bounds for the function $\pi(x)$ (see [2]):

$$L(x) = \frac{x}{\ln x}\Big(1 + \frac{1}{\ln x} + \frac{1.8}{\ln^2 x}\Big) \le \pi(x) \qquad (x \ge 32299), \tag{1.1}$$

and

$$\pi(x) \le U(x) = \frac{x}{\ln x}\Big(1 + \frac{1}{\ln x} + \frac{2.51}{\ln^2 x}\Big) \qquad (x \ge 355991). \tag{1.2}$$

2. Main Result

Lemma 2.1. *For every $n \ge 2$, we have*

$$\frac{n^2}{2\ln n} + 4 - \frac{9}{\ln 9} - \sum_{k=3}^{n-1} \frac{\ln^2 k}{\ln\ln k} < \frac{n^2}{2\ln n}\left(1 + \frac{1}{2\ln n} + \frac{9}{20\ln^2 n}\right).$$

Proof. For every $n \ge 2$, consider the following inequality

$$\frac{n^2}{4\ln^2 n} + \frac{9n^2}{40\ln^3 n} + \sum_{k=3}^{n-1} \frac{\ln^2 k}{\ln\ln k} > 4 - \frac{9}{\ln 9}.$$

Note that the left member of it, is positive and the right member is negative. So, clearly it holds for every $n \ge 2$. □

Lemma 2.2. *For every $n \ge 180$, we have*

$$\frac{n^2}{2\ln n} + 4 - \frac{9}{\ln 9} - \sum_{k=3}^{n-1} \frac{\ln^2 k}{\ln\ln k} < \pi(n^2).$$

Proof. Putting $x = n^2$ in (1.1), for $n \ge 180 = \lceil\sqrt{32299}\rceil$ we obtain

$$\frac{n^2}{2\ln n}\left(1 + \frac{1}{2\ln n} + \frac{9}{20\ln^2 n}\right) < \pi(n^2).$$

Considering this, with previous lemma, completes the proof. □

Theorem 2.3. *For many infinity positive integer values of n, the following inequality holds*

$$\left\lfloor \frac{1}{2}\left(\frac{(n+1)^2}{\ln(n+1)} - \frac{n^2}{\ln n}\right) - \frac{\ln^2 n}{\ln\ln n}\right\rfloor \le \pi\big((n+1)^2\big) - \pi(n^2).$$

Proof. Reform the truth of lemma 2.2, as follows:

$$\frac{1}{2}\left(\frac{n^2}{\ln n} - \frac{3^2}{\ln 3}\right) - \sum_{k=3}^{n-1} \frac{\ln^2 k}{\ln\ln k} < \pi(n^2) - \pi(3^2).$$

This inequality yields the following one:

$$\sum_{k=3}^{n-1} \left\lfloor \frac{1}{2}\left(\frac{(k+1)^2}{\ln(k+1)} - \frac{k^2}{\ln k}\right) - \frac{\ln^2 k}{\ln\ln k}\right\rfloor < \sum_{k=3}^{n-1} \pi\big((k+1)^2\big) - \pi(k^2),$$

which holds for all $n \geq 180$. Now, we note that terms under summations, in both sides are non-negative integers and this completes the proof[1]. □

However, this challenge was unsuccessful for proving the relation

$$\{n|\ \left(n^2,(n+1)^2\right)\cap\mathbb{P}\neq\phi\}=\mathbb{N},$$

but it seems that it can be useful for improving it. To see this, let

$$g(n)=\#\big\{t\mid t\in\mathbb{N},\ t\leq n,\ \mathbb{P}\cap\left(t^2,(t+1)^2\right)\neq\phi\big\}.$$

Clearly, $\lim_{n\to\infty} g(n)=\infty$ and $g(n)\leq n$. Note that $g(n)=n$ is above mentioned open problem. A lower bound for $g(n)$ is the following bound, which we can yield by considering previous theorem for every $n\geq 597$;

$$g(n)\geq M(n),$$

in which

$$M(n)=\max_m\left\{\sum_{k=597}^{n}\left\lfloor\frac{1}{2}\left(\frac{(k+1)^2}{\ln(k+1)}-\frac{k^2}{\ln k}\right)-\frac{\ln^2 k}{\ln\ln k}\right\rfloor\leq\sum_{k=m}^{n}U\big((k+1)^2\big)-L(k^2)\right\}.$$

Clearly, if $n\to\infty$, then we have

$$M(n)=O(n).$$

Also, we have the following conjecture on the size of $M(n)$:

Conjecture 3. For every $\epsilon>0$ there exists $n_\epsilon\in\mathbb{N}$ such that for all $n>n_\epsilon$ we have

$$M(n)>(1-\epsilon)n.$$

References

[1] H. Davenport, Multiplicative Number Theory (Second Edition), Springer-Verlag, 1980.
[2] P. Dusart, Inégalités explicites pour $\psi(X)$, $\theta(X)$, $\pi(X)$ et les nombres premiers, *C. R. Math. Acad. Sci. Soc. R. Can.* **21** (1999), no. 2, 53–59.

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[1]In fact we can show that if a_n and b_n are two non-negative integer sequences, with $\sum_{n=n_0}^{N} a_n<\sum_{n=n_0}^{N} b_n$, then we have $\#\{n|\ a_n\leq b_n\}=\aleph_0$.