AN EQUIVALENT FOR PRIME NUMBER THEOREM

MEHDI HASSANI

ABSTRACT. In this short research report, we show that $\Psi(p_n) \sim \log n$, when $n \to \infty$ is equivalent with Prime Number Theorem, in which $\Psi(x) = \frac{d}{dx}\log\Gamma(x)$ is digamma function.

1. Introduction and Main Result

As usual, let $\mathbb{P}$ be the set of all primes and $\pi(x) = \#\mathbb{P} \cap [2, x]$. Also, for $x > 0$ define digamma function $\Psi(x)$ to be $\Psi(x) = \frac{d}{dx}\log\Gamma(x)$, in which $\Gamma(x) = \int_0^\infty e^{-t}t^{x-1}dt$ is well-known gamma function [1].
In [4], it is shown that for every $x \geq 3299$, we have

$$\frac{x}{\Psi(x) - A} < \pi(x) < \frac{x}{\Psi(x) - B},$$

in which $A = \frac{3298}{3299} - \frac{1}{4\log 3299} \approx 0.969$ and $B = 2 + \frac{151}{100\log 7} - \gamma \approx 2.199$. So, for every $x \geq 3299$, we obtain

$$\frac{x}{\pi(x)} + A < \Psi(x) < \frac{x}{\pi(x)} + B,$$

and by putting $x = p_n$, n^{th} prime, for $n \geq 463$ we yield that

$$\frac{p_n}{n} + A < \Psi(p_n) < \frac{p_n}{n} + B. \tag{1.1}$$

In other hand, we have [3] the following sharp bounds for p_n, which holds for every $n \geq 27076$

$$\log n + \log_2 n - 1 + \frac{\log_2 n - 2.25}{\log n} \leq \frac{p_n}{n} \leq \log n + \log_2 n - 1 + \frac{\log_2 n - 1.8}{\log n},$$

in which $\log_2 = \log\log$ and base of all logarithms is e. Considering this inequality with (1.1), for every $n \geq 27076$ we obtain

$$\log n + \log_2 n + A - 1 + \frac{\log_2 n - 2.25}{\log n} < \Psi(p_n) < \log n + \log_2 n + B - 1 + \frac{\log_2 n - 1.8}{\log n}.$$

This inequality is very strong form of an equivalent of Prime Number Theorem (PNT), which asserts $\pi(x) \sim \frac{x}{\log x}$ and is equivalent with $p_n \sim n\log n$ (see [2]). In this note, we show that $\Psi(p_n) \sim \log n$, when $n \to \infty$ is another equivalent for PNT. To do this, first suppose PNT. Thus, we have $p_n = n\log n + o(n\log n)$. Also, (1.1) yields that $\Psi(p_n) = \frac{p_n}{n} + O(1)$. Therefore, we have

$$\Psi(p_n) = \frac{n\log n + o(n\log n)}{n} + O(1) = \log n + o(\log n).$$

1991 *Mathematics Subject Classification.* 11A41, 26D15, 33B15.
Key words and phrases. Primes, Prime Number Theorem, Gamma function, Digamma function, Inequalities.

Conversely, suppose $\Psi(p_n) = \log n + o(\log n)$. By solving (1.1) according to p_n, we obtain

$$n\Psi(p_n) - Bn < p_n < n\Psi(p_n) - An.$$

Therefore, we have

$$p_n = n\Psi(p_n) + O(n) = n\big(\log n + o(\log n)\big) + O(n) = n\log n + o(n\log n),$$

which, this is PNT.

References

[1] M. Abramowitz and I.A. Stegun, *HANDBOOK OF MATHEMATICAL FUNCTIONS: with Formulas, Graphs, and Mthematical Tables*, Dover Publications, 1972.

[2] Tom. M. Apostol, Introduction to Analytic Number Theory, Springer 2000.

[3] P.Dusart, Inégalités explicites pour $\psi(X)$, $\theta(X)$, $\pi(X)$ et les nombres premiers, *C. R. Math. Acad. Sci. Soc. R. Can.*, 21 (1999), no. 2, 53-59.

[4] M. Hassani, Approximation of $\pi(x)$ by $\Psi(x)$, *RGMIA Research Report Collection*, **8**(2), Article 2, 2005.

Institute for Advanced, Studies in Basic Sciences, P.O. Box 45195-1159, Zanjan, Iran.

E-mail address: `mhassani@iasbs.ac.ir`