SOME REFINEMENTS OF RELATIVE INFORMATION INEQUALITY

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Abstract. In this note, using some refinements of Jensen's discrete inequality, we give some new refinements of Kullback-Leibler's relative information inequality.

1. Introduction

Let C be a convex subset of a real vector space, $x_1, \cdots, x_n \in C$, and $\varphi : C \to \mathbb{R}$ a convex mapping. Also, let $\mu = (\mu_1, \cdots, \mu_m)$ and $\lambda = (\lambda_1, \cdots, \lambda_n)$ be two probability measures; i.e. $\mu_i,\ \lambda_j \geq 0$ $(1 \leq i \leq m,\ 1 \leq j \leq n)$ with

$$\sum_{i=1}^{m} \mu_i = 1 \qquad \text{and} \qquad \sum_{j=1}^{n} \lambda_j = 1.$$

By a (discrete separately) weight function (with respect to μ and λ), we always mean a mapping

$$\omega : \{(i,j) :\ 1 \leq i \leq m,\ 1 \leq j \leq n\} \to [0, \infty),$$

such that

$$\sum_{i=1}^{m} \omega(i,j)\mu_i = 1 \qquad (j = 1, \cdots, n),$$

and

$$\sum_{j=1}^{n} \omega(i,j)\lambda_j = 1 \qquad (i = 1, \cdots, m).$$

For example, if $u = (u_1, \cdots, u_m)$ and $v = (v_1, \cdots, v_n)$ with $\|u\| = (\sum_{i=1}^{m} u_i^2)^{1/2} \leq 1$ and $\|v\| = (\sum_{j=1}^{n} v_j^2)^{1/2} \leq 1$ belong to $\mu^{\perp}$ and $\lambda^{\perp}$ respectively, then the function ω with $\omega(i,j) = 1 + u_i v_j$ $(1 \leq i \leq m,\ 1 \leq j \leq n)$ is a weight function.
In [2] the following refinement of discrete Jensen's inequality is established:
If ω_1 and ω_2 are two weight functions, then we have
(1.1)

$$\varphi\left(\sum_{j=1}^{n} \lambda_j x_j\right) \leq \sum_{i=1}^{m} \mu_i A\left(\varphi; \sum_{j=1}^{n} \omega_1(i,j)\lambda_j x_j, \sum_{j=1}^{n} \omega_2(i,j)\lambda_j x_j\right) \leq \sum_{j=1}^{n} \lambda_j \varphi(x_j),$$

where the arithmetic mean A is defined for an integrable function f over an interval with end points a and b, by

$$A(f; a, b) = \frac{1}{b-a}\int_a^b f(x)dx. \tag{1.2}$$

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(*We set* $A(f;a,a)=f(a)$.)
In the following section, using this fact, we refine the (Kullback-Leibler's relative) information inequality, and in particular, we obtain some inequalities concerning special means.

2. Main Results

Let $p=(p_1,\cdots,p_n)$ and $q=(q_1,\cdots,q_n)$ be such that $p_j,q_j>0$ $(1\le j\le n)$ with $\sum_{j=1}^n p_j=\sum_{j=1}^n q_j=1$. The Kullback-Leibler's relative information $D(p||q)$ is defined by

$$D(p||q)=\sum_{j=1}^n p_j \ln\frac{p_j}{q_j}. \tag{2.1}$$

The information inequality [1] is

$$D(p||q)\ge 0. \tag{2.2}$$

In this section, using the refinement of discrete Jensen's inequality described above, we give some new refinements of information inequality (2.2). In particular, we get some interesting inequalities between various means of p_j's and q_j's, which are difficult to handle them directly.

Theorem 2.1. *With the above assumptions, we have*

$$D(p||q)\ge \ln\prod_{i=1}^m I\left(\sum_{j=1}^n \omega_1(i,j)q_j,\sum_{j=1}^n \omega_2(i,j)q_j\right)^{-\mu_i}\ge 0, \tag{2.3}$$

where the identric mean I is defined for each $a,b>0$ by

$$I(a,b)=\begin{cases} a & \text{if } a=b,\\ \frac{1}{e}\left(\frac{b^b}{a^a}\right)^{\frac{1}{b-a}} & \text{if } a\ne b.\end{cases}$$

In particular,

$$\prod_{i=1}^n q_i^{p_i}\le\prod_{i=1}^n I(p_i,q_i)^{p_i}\le\prod_{i=1}^n p_i^{p_i}. \tag{2.4}$$

Proof. The function $\varphi(x)=-\ln x$ is convex on $(0,+\infty)$. So, letting $\lambda_j=p_j$ and $x_j=\frac{q_j}{p_j}$ $(1\le j\le n)$ in (1.1), and taking into account that

$$A(-\ln;a,b)=-\ln I(a,b) \qquad (a,b>0),$$

we get (2.3).
The inequalities in (2.4) follow from (2.3), by taking $m=n$, $\mu_i=p_i$, $\omega_1(i,j)=1$, $\omega_2(i,j)=\frac{\delta_{ij}}{p_j}$ $(i,j=1,\cdots,n)$, and considering

$$I(ac,bc)=cI(a,b) \qquad (a,b,c>0).$$

Theorem 2.2. *With the above assumptions, we have*
(2.5)
$$D(p||q) \geq \ln \sqrt{\prod_{i=1}^{m} I\left(\left[\sum_{j=1}^{n} \omega_1(i,j)p_j\right]^2, \left[\sum_{j=1}^{n} \omega_2(i,j)p_j\right]^2\right)^{\sum_{j=1}^{n} \frac{\omega_1(i,j)+\omega_2(i,j)}{2} p_j \mu_i}} \geq 0.$$

In particular,

$$\prod_{i=1}^{n} \left(\frac{p_i}{q_i}\right)^{p_i} \geq \sqrt{\prod_{i=1}^{n} I\left(1, \frac{p_i^2}{q_i^2}\right)^{\frac{p_i+q_i}{2}}} \geq 1. \tag{2.6}$$

Proof. The function $\varphi(x) = x \ln x$ is convex on $(0, +\infty)$. So, letting $\lambda_j = q_j$ and $x_j = \frac{p_j}{q_j}$ $(j = 1, \cdots, n)$ in (1.1), and taking into account that

$$A(\varphi; a, b) = \frac{a+b}{4} \ln I(a^2, b^2) \qquad (a, b > 0),$$

we get (2.5).
The inequalities in (2.6) follow from (2.5) by taking $m = n,\ \mu_i = q_i,\ \omega_1(i,j) = 1,\ \omega_2(i,j) = \frac{\delta_{ij}}{q_j}$ $(i, j = 1, \cdots, n)$.

Theorem 2.3. *With the above assumptions, we have*

$$\prod_{j=1}^{n} \left(\frac{q_j}{p_j}\right)^{p_j} \leq \sum_{i=1}^{m} \mu_i L\left(\prod_{j=1}^{n} \left(\frac{q_j}{p_j}\right)^{\omega_1(i,j)p_j}, \prod_{j=1}^{n} \left(\frac{q_j}{p_j}\right)^{\omega_2(i,j)p_j}\right) \leq 1, \tag{2.7}$$

where the logarithmic mean L is defined for each $a, b > 0$, by

$$L(a,b) = \begin{cases} a & \text{if } a = b, \\ \frac{b-a}{\ln b - \ln a} & \text{if } a \neq b. \end{cases}$$

In particular,

$$\prod_{j=1}^{n} \left(\frac{q_j}{p_j}\right)^{p_j} \leq \sum_{i=1}^{n} p_i L\left(\prod_{j=1}^{n} \left(\frac{q_j}{p_j}\right)^{p_j}, \frac{q_i}{p_i}\right) \leq 1. \tag{2.8}$$

Proof. The function $\varphi(x) = e^x$ is convex on $\mathbb{R}$ and we have

$$A(\varphi; a, b) = L(e^a, e^b) \qquad (a, b \in \mathbb{R}).$$

Therefore, taking $x_j = \ln \frac{q_j}{p_j}$ and $\lambda_j = p_j$ $(1 \leq j \leq n)$ in (1.1), we get

$$\begin{aligned} \exp\left(\sum_{j=1}^{n} p_j \ln \frac{q_j}{p_j}\right) &\leq \sum_{i=1}^{m} \mu_i L\left(e^{\sum_{j=1}^{n} \omega_1(i,j)p_j \ln \frac{q_j}{p_j}}, e^{\sum_{j=1}^{n} \omega_2(i,j)p_j \ln \frac{q_j}{p_j}}\right) \\ &\leq \sum_{j=1}^{n} p_j \exp\left(\ln \frac{q_j}{p_j}\right), \end{aligned}$$

which yield (2.7).
Now, letting $m = n,\ \mu_i = p_i,\ \omega_1(i,j) = 1$ and $\omega_2(i,j) = \frac{\delta_{ij}}{p_j}$ $(i, j = 1, \cdots, n)$ in (2.7), we get (2.8).

Remark 2.1. If we set $q_j = \frac{1}{n}$ $(1 \leq j \leq n)$, the relative information inequality (2.2), yields the entropy of the probability distribution inequality

$$H(p_1, \cdots, p_n) := -\sum_{i=1}^{n} p_i \ln p_i \leq \ln n.$$

This inequality has been refined in [3] as

$$\frac{1}{n} \leq \sqrt{\prod_{i=1}^{n} I\left(\left[\sum_{j=1}^{n} b_{ij} p_j\right]^2, \left[\sum_{j=1}^{n} c_{ij} p_j\right]^2\right)^{\sum_{j=1}^{n} \frac{b_{ij}+c_{ij}}{2} p_j}} \leq \prod_{i=1}^{n} p_i^{p_i}, \tag{2.9}$$

where $B = [b_{ij}]$ and $C = [c_{ij}]$ are two $n \times n$ double stochastic matrices [2]. It is easy to see that (2.9) is an special case of (2.5), taking $m = n,\ \mu_i = q_i = \frac{1}{n},\ \omega_1(i,j) = nb_{ij},\ \omega_2(i,j) = nc_{ij}\ (i,j = 1, \cdots, n)$.

Remark 2.2. If we change the roles of p_i's and q_i's with each other in (2.4) and (2.6), and multiply them correspondingly, we get

$$\sqrt{\prod_{i=1}^{n} p_i^{q_i} \prod_{i=1}^{n} q_i^{p_i}} \leq \prod_{i=1}^{n} I(p_i, q_i)^{\frac{p_i+q_i}{2}} \leq \sqrt{\prod_{i=1}^{n} p_i^{p_i} \prod_{i=1}^{n} q_i^{q_i}} \tag{2.10}$$

and

$$1 \leq \prod_{i=1}^{n} \left(\frac{I(p_i^2, q_i^2)}{p_i q_i}\right)^{\frac{p_i+q_i}{2}} \leq \frac{\prod_{i=1}^{n} p_i^{p_i} \prod_{i=1}^{n} q_i^{q_i}}{\prod_{i=1}^{n} p_i^{q_i} \prod_{i=1}^{n} q_i^{p_i}}, \tag{2.11}$$

which are symmetric with respect to p_i's and q_i's.

References

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