On Modified Hyperperfect Numbers

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Abstract. We introduce the so-called modified hyperperfect numbers, and completely determine their form in the case of classical divisors, unitary, bi-unitary, and e-divisors, respectively.

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1. Introduction.

Let d be a positive divisor of the integer $n > 1$. If $\left(d, \frac{n}{d}\right) = 1$, then d is called a unitary divisor of n. If the greatest common unitary divisor of d and n/d is 1, then d is called a bi-unitary divisor of n. If $n = p_1^{a_1}...p_r^{a_r} > 1$ is the prime factorization of n, a divisor d of n is called an exponential divisor (or e-divisor, for short), if $d = p_1^{b_1}...p_r^{b_r} > 1$ with $b_i|a_i$ $\left(i = \overline{1, r}\right)$. For the history of these notions, as well as the connected arithmetical functions, see e.g. [1], [2]. In what follows $\sigma(n), \sigma^*(n), \sigma^{**}(n), \sigma_e(n)$ will denote the sum of divisors, -unitary divisors, -bi-unitary divisors, and e-divisors, respectively. It is well-known that a positive integer m is called n-hyperperfect (HP-for short), if

(1). $m = 1 + n[\sigma(m) - m - 1]$

For $n = 1$ one has $\sigma(m) = 2m$, i.e. the 1-HP numbers coincide with the classical perfect numbers. For results on HP-numbers, see [2].

Let $f : N^* \to N^* = \{1, 2, ...\}$ be an arithmetical function. Then m will be called f-n-hyperperfect number, if

(2). $m = 1 + n[f(m) - m - 1]$

for some integer $n \geq 1$. For $f(m) = \sigma(m)$ one obtains the HP-numbers, while for $f(m) = \sigma^*(m)$, we get the unitary hyperperfect numbers (UHP) introduced by P. Hagis [2]. When $f(m) = \sigma^{**}(m)$, we get the bi-unitary hyperperfect numbers (BHP), introduced by the first author ([3]). For $f(m) = \sigma_e(m)$, we get the e-hyperperfect numbers (e-HP), introduced also by the first author [4].

2. Modified hyperperfect numbers.

In what follows, m will be called a modified f-n-hyperperfect number, if

(3). $m = n[f(m) - m]$

For $f(m) = \sigma(m)$, we get the modified hyperperfect numbers (MHP). Since for $n = 1$ one has in (3) $f(m) = 2m$, one obtains again a generalization of f-perfect numbers. First we prove:

Theorem 1. All MHP numbers are the classical perfect numbers, as well as the prime numbers.

Proof. Since (3) implies $n|m$, put $m = kn$, giving $k = f(kn) - kn$, so

(3'). $f(kn) = k(n + 1)$

For $f \equiv \sigma$ this gives

(4). $\sigma(kn) = k(n+1)$

For $n = 1$, (4) gives $\sigma(k) = 2k$, so $m = k$ is the classical perfect number.

For $k = 1$, relation (4) implies $\sigma(n) = n+1$, which is possible only for $n = p$ (prime), since $\sigma(n) \geq n+1$, with equality if n has only two distinct divisors - namely 1 and n -, so n is a prime. Thus $m = p$ is a modified p-hyperperfect number.

Assume now that, $k > 1, n > 1$ in (4). Then it is well known that $\sigma(kn) > k\sigma(n)$ (see e.g. [2]). Since $\sigma(n) \geq n+1$ for all $n > 1$, we can infern that $\sigma(kn) > k(n+1)$, in contradiction with (4).

For the case of unitary divisors, one can state:

Theorem 2. All UMHP-numbers are the unitary perfect numbers, as well as, the prime powers.

Proof. (3') now becomes

(5). $\sigma^*(kn) = k(n+1)$

For $n = 1$ we get $\sigma^*(k) = 2k$, i.e. k is a unitary perfect number.

For $k = 1$ we get $\sigma^*(n) = n+1$, which is true only for $n = p^a$ (prime power), by $\sigma^*(n) = \prod_{p^a \| n} (p^a + 1)$.

Let us now assume that $n, k > 1$. Since $k(n+1) = kn + k$, and k is not only a divisor, but a unitary one of kn, one can write $\left(k, \frac{nk}{k}\right) = 1$, i.e. $(k, n) = 1$. But then, σ^* being multiplicative, $\sigma^*(kn) = \sigma^*(k)\sigma^*(n) \geq (k+1)(n+1) > k(n+1)$ for $k > 1, n > 1$. This contradicts (5), so Theorem 2 is proved.

For bi-unitary divisors we can state:

Theorem 3. All BMHP-numbers are 6, 60, 90; as well as all primes or squares of primes.

Proof. (3') now is

(6). $\sigma^{**}(kn) = k(n+1)$

Where $\sigma^{**}(n)$ denotes the sum of bi-unitary divisors of n. For $n = 1$ we get $\sigma^{**}(k) = 2k$, so by a result of Ch. Wall (see [2]) one can write $k \in \{6, 60, 90\}$.

For $k = 1$ we get $\sigma^{**}(n) = n+1$, which is possible only for $n = p$ or $n = p^2$ (p=prime). This is well-known, but we note that it follows also from $\sigma^*(p^a) = \sigma(p^a)$, if a is odd (p=prime), $\sigma^{**}(p^a) = \sigma(p^a) - p^{a/2}$ if a is even; and the multiplicativity of σ^{**}.

Let now $k > 1, n > 1$. Then $kn \neq k, kn \neq n$, and $\left(k, \frac{nk}{k}\right)_* = (k, n)_* = 1$ where $(k, n)_*$ denotes the greatest common unitary divisors of k and n. Since $k \neq n$, by $(k, n)_* = 1$, and n is also a divisor of n, but not a bi-unitary one, by (5) (i.e. $\sigma^{**}(kn) = kn + k$ - which means that the only bi-unitary divisors of kn are kn and k). But then $\left(n, \frac{kn}{n}\right)_* \neq 1$, i.e. $(n, k)_* \neq 1$, in contradiction with $(k, n)_* = 1$.

Finally, the case of e-divisors is contained in:

Theorem 4. All modified exponentially n-hyperperfect numbers m are given by $m = kn$, where $k = p_1 p_2 ... p_r$, $n = p_1^{q-1} p_2 ... p_r$, with $p_1, p_2, ..., p_r$ distinct primes, and q an arbitrary primes; as well as the e-perfect numbers.

Proof. We have to study the equation:

(7). $\sigma_e(kn) = k(n+1)$

For $n = 1$ we have $\sigma_e(k) = 2k$, i.e. the e-perfect numbers.

For $k = 1$ we get $\sigma_e(n) = n+1$. For $n =$ squarefree, one has $\sigma_e(n) = n$, while for $n \neq$ squarefree, by the above lemma, $\sigma_e(n) > n + 1$, giving a contradiction.

Lemma. If n is not squarefree, $n > 1$ then $\sigma_e(n) \geq n + n/q^{a-1}$, where $q^a \parallel n$ and $a \geq 2$. There is equality only for $p_1...p_r q^p$ with $p_1, ..., p_r, q$ distinct primes, and p an arbitrary prime.

Proof. Let $n = p_1^{a_1}...p_r^{a_r}$, where $(\exists)\, a \in \{a_1, ...a_r\}$ with $a \geq 2$. Thus $\sigma_e(n) \geq p_1^{a_1}.../...p_r^{a_r}(q^a + q) = n + \underbrace{p_1^{a_1}...q...p_r^{a_r}}_{n/q^{a-1}}$ since $\sigma_e(q^a) \geq q^1 + q^a$ with equality only if $a =$prime, while $\sigma_e\left(p^b\right) \geq p^b$, with equality only for $b = 1$.

Corollary. $\sigma_e(n) \geq n + \gamma(n)$ for $n \neq$ squarefree, where $\gamma(n) = \prod_{p|n} p =$ product of distinct prime divisors of n.

a). Now, suppose that $(n, k) = 1$. Since σ_e is multiplicative, (7) becomes $\sigma_e(n)\sigma_e(k) = k(n+1)$. If $k > 1$ is squarefree, then $\sigma_e(k) = k$, so this is $\sigma_e(n) = n + 1$, which is impossible. If k is not squarefree, but n is squarefree, then $\sigma_e(n) = n$, so (7) becomes $n\sigma_e(k) = k(n+1)$. Since $(n, k) = 1$ and $(n, n+1) = 1$, this is again impossible.

By summarizing, if $(n, k) = 1$ for $n > 1, k > 1$, the equation is unsolvable.

b). Let $(n, k) > 1$. Writing $n = p_1^{a_1}...p_r^{a_r} q_1^{b_1}...q_s^{b_s}$, $k = p_1^{a'_1}...p_r^{a'_r}\gamma_1^{c_1}...\gamma_t^{c_t}$, where p_i, q_j, γ_k are distinct prime, and a_i, b_j, c_k are nonnegative integers ($1 \leq i \leq r$, $1 \leq j \leq s$, $1 \leq k \leq t$). Since by (7) written in the form $\sigma_e(kn) = kn + k - k$ is an e-divisor of $nk = p_1^{a_1+a'_1}...p_r^{a_r+a'_r} \cdot q_1^{b_1}...q_s^{b_s} \cdot \gamma_1^{c_1}...\gamma_t^{c_t}$, we must have $b_1 = ... = b_s = 0$. Also $a'_1|(a_1 + a'_1), ..., a'_r|(a_r + a'_r)$, i.e. $a_1 = (m_1 - 1)a'_1, ..., a_r = (m_r - 1)a'_r$, with m_i $(1 \leq i \leq r)$ positive integers.

We note that, since $a_1 \geq 1, ..., a_r \geq 1$, we have $m_1 > 1, ..., m_r > 1$. Since $\gamma_1^{c_1} \neq k$ is also an e-divisor of nk, we must have $c_1 = 0$. Similarly, $c_2 = ... = c_r = 0$. Thus, $n = p_1^{(m_1-1)a'_1}...p_r^{(m_r-1)a'_r}$, $k = p_1^{a'_1}...p_r^{a'_r}$; $nk = p_1^{m_1 a'_1}...p_r^{m_r a'_r}$. Remark that by $m_1 > 1$, if one assumes $a'_1 > 1$, then $1|m_1 a'_1$ implies that $p_1...p_r$ is also an e-divisor of nk, with $p_1...p_k \neq k$. Thus we must have necessarily $a'_1 = ... = a'_r = 1$, so $k = p_1...p_r$ and $nk = p_1^{m_1}...p_r^{m_r}$. Since $n > 1$, at least one of $m_1, ..., m_r$ is > 1. Put $m_1 > 1$. Then, if at least one of $m_2, ..., m_r$ is > 1, then $\mathrm{p}_1 p_2^{m_2}...p_r^{m_r} \neq k$ is another e-divisor of nk. If $m_1 > 1$ is not a prime, then m_1 can have also a divisor $1 < a < m_1$ so $p_1^a p_2...p_r$ will be another e-divisor, contradiction. Thus $a = q =$prime, which finishes the proof of Theorem 4.

For results and/or open problems on perfect, unitary perfect, e-perfect numbers; as well as on hyperperfect or unitary hyperperfect numbers, see the monographs [1], [2].

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