APPROXIMATION OF THE PRODUCT $p_1p_2\cdots p_n$

MEHDI HASSANI

Abstract. In this research report, we find some lower and upper bounds for the product $p_1p_2\cdots p_n$.

1. Introduction and Approximation of the Product $p_1p_2\cdots p_n$

As usual, let p_n be the n^{th} prime. The Mondl's inequality (see [1] and [8]) asserts that for every $n \geq 9$, we have $\sum_{i=1}^{n} p_i < \frac{n}{2}p_n$. In [2], one has get the following refinement of this inequality:

$$\sum_{i=1}^{n} p_i < \frac{n}{2}p_n - 0.01659n^2 \qquad (n \geq 9). \tag{1.1}$$

Considering the AGM Inequality (see [7]) and (1.1), for every $n \geq 9$, we obtain:

$$p_1p_2\cdots p_n < \Big(\frac{p_n}{2} - 0.01659n\Big)^n \qquad (n \geq 9). \tag{1.2}$$

Note that (1.2) holds also for $5 \leq n \leq 8$. This yields an upper bound for the product $p_1p_2\cdots p_n$. About lower bound, one can get a trivial one for that product, using Euclid's proof of infinity of primes; Letting $E_n = p_1p_2\cdots p_n - 1$, for every $n \geq 2$, it is clear that $p_n < E_n$. So, if $p_n < E_n < p_{n+1}$, then E_n should has a prime factor among $p_1, p_2, \cdots, p_n$, which isn't possible. Thus, $E_n \geq p_{n+1}$, and therefore, for every $n \geq 2$, we have:

$$p_1p_2\cdots p_n > p_{n+1}.$$

In 1957 in [5], Bonse used elementary methods to show that:

$$p_1p_2\cdots p_n > p_{n+1}^2 \qquad (n \geq 4),$$

and

$$p_1p_2\cdots p_n > p_{n+1}^3 \qquad (n \geq 5).$$

In 1960 Pósa [4] proved that for every $k > 1$, there exists an n_k, such that for all $n \geq n_k$, we have:

$$p_1p_2\cdots p_n > p_{n+1}^k.$$

In 1988, J. Sandór found some inequalities of similar type; For example he showed that for every $n \geq 24$, we have:

$$p_1p_2\cdots p_n > p_{n+5}^2 + p_{[\frac{n}{2}]}^2.$$

In 2000, Panaitopol [3] showed that in Pósa's result, we can get $n_k = 2k$. More precisely, he proved that for every $n \geq 2$, we have:

$$p_1p_2\cdots p_n > p_{n+1}^{n-\pi(n)},$$

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in which $\pi(x)$ = the number of primes $\leq x$. In this note, we refine Panaitopol's result by proving

$$p_1 p_2 \cdots p_n > p_{n+1}^{(1-\frac{1}{\log n})(n-\pi(n))} \qquad (n \geq 101). \tag{1.3}$$

During proofs, we will need some known results, which we review them briefly; we have the following bound [1]:

$$\pi(x) \geq \frac{x}{\log x}\left(1 + \frac{1}{\log x}\right) \qquad (x \geq 599). \tag{1.4}$$

For every $n \geq 53$, we have [3]:

$$\log p_{n+1} < \log n + \log\log n + \frac{\log\log n - 0.4}{\log n}. \tag{1.5}$$

Also, for every $n \geq 3$, we have [6]:

$$\theta(p_n) > n\left(\log n + \log\log n - 1 + \frac{\log\log n - 2.1454}{\log n}\right), \tag{1.6}$$

in which $\theta(x) = \sum_{p\leq x} \log p$. Specially, $\theta(p_n) = \log(p_1 p_2 \cdots p_n)$ and this will act as a key for approximating $p_1 p_2 \cdots p_n$. Finally, just for insisting, we note that base of all logarithms are e. To prove (1.3), considering (1.4), (1.5) and (1.6), it is enough to prove that:

$$\left(1 - \frac{1-\frac{1}{\log n}}{\log n} - \frac{1-\frac{1}{\log n}}{\log^2 n}\right)\left(\log n + \log\log n + \frac{\log\log n - 0.4}{\log n}\right)$$
$$< \log n + \log\log n - 1 + \frac{\log\log n - 2.1454}{\log n} \qquad (n \geq 599),$$

which putting $x = \log n$, is equivalent with:

$$\frac{1.7454x^3 + 1.4x^2 - 0.4}{x^3 + x^2 - x - 1} < \log x \qquad x \geq \log 599,$$

and trivially, this holds true; because for $x \geq \log 599$ we have $\frac{1.7454x^3+1.4x^2-0.4}{x^3+x^2-x-1} < 1.7454$, and $1.85 < \log x$. Therefore, we yield (1.3) for all $n \geq 599$. For $101 \leq n \leq 598$, we get it by a computer.

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Institute for Advanced, Studies in Basic Sciences, P.O. Box 45195-1159, Zanjan, Iran.
E-mail address: mhassani@iasbs.ac.ir