APPROXIMATION OF $\pi(x)$ BY $\Psi(x)$

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Abstract. In this paper we find some lower and upper bounds of the form $\frac{n}{H_n - c}$ for the function $\pi(n)$, in which $H_n = \sum_{k=1}^{n} \frac{1}{k}$. Then, considering $H(x) = \Psi(x+1) + \gamma$ as generalization of H_n, in which $\Psi(x) = \frac{d}{dx} \log \Gamma(x)$ and γ is Euler constant, we find some lower and upper bounds of the form $\frac{x}{\Psi(x) - c}$ for the function $\pi(x)$.

1. Introduction

As usual, let $\mathbb{P}$ be the set of all primes and $\pi(x) = \#\mathbb{P} \cap [2, x]$. If $H_n = \sum_{k=1}^{n} \frac{1}{k}$, then easily we have

$$\gamma + \log n < H_n < 1 + \log n \qquad (n > 1), \tag{1.1}$$

in which γ is Euler constant. So, $H_n = \log n + O(1)$ and considering the prime number theorem [1], we obtain

$$\pi(n) = \frac{n}{H_n + O(1)} + o\left(\frac{n}{\log n}\right).$$

Thus, comparing $\frac{n}{H_n + O(1)}$ with $\pi(n)$ seems to be a nice problem. In 1959, L. Locker-Ernst [3] affirms that $\frac{n}{H_n - \frac{3}{2}}$ is very close to $\pi(n)$ and in 1999, L. Panaitopol [5] proved that for $n \geq 1429$ it is actually a lower bound for $\pi(n)$.
In this paper we improve Panaitopol's result by proving $\frac{n}{H_n - a} < \pi(n)$ for every $n \geq 3299$, in which $a \approx 1.546356705$. Also, we find same upper bound for $\pi(n)$. Then we consider generalization of H_n as a real value function which has been studied by J. Sándor [6] in 1988; for $x > 0$ let $\Psi(x) = \frac{d}{dx} \log \Gamma(x)$, in which $\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt$ is well-known gamma function. Since $\Gamma(x+1) = x\Gamma(x)$ and $\Gamma(1) = -\gamma$, we have $H_n = \Psi(n+1) + \gamma$, and this relation led him to define

$$\begin{cases} H : (0, 1) \longrightarrow \mathbb{R}, \\ H(x) = \Psi(x+1) + \gamma, \end{cases} \tag{1.2}$$

as a natural generalization of H_n, and more naturally, it motivated us to find some bounds for $\pi(x)$ concerning $\Psi(x)$. In our proofs, we use the obvious relation

$$\Psi(x+1) = \Psi(x) + \frac{1}{x}. \tag{1.3}$$

Also, we need some bounds of the form $\frac{x}{\log x - 1 - \frac{c}{\log x}}$, which we yield them by using the following known sharp bounds [2] for $\pi(x)$

$$\frac{x}{\log x}\left(1 + \frac{1}{\log x} + \frac{1.8}{\log^2 x}\right) \leq \pi(x) \qquad (x \geq 32299), \tag{1.4}$$

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and

$$\pi(x) \leq \frac{x}{\log x}\Big(1+\frac{1}{\log x}+\frac{2.51}{\log^2 x}\Big) \qquad (x \geq 355991). \tag{1.5}$$

2. Bounds of the form $\frac{x}{\log x - 1 - \frac{c}{\log x}}$

Lower Bounds. We are going to find suitable values of a, in which $\frac{x}{\log x-1-\frac{a}{\log x}} \leq \pi(x)$. Considering (1.4) and letting $y = \log x$, we should study the inequality $\frac{1}{y-1-\frac{a}{y}} \leq \frac{1}{y}(1+\frac{1}{y}+\frac{9}{5y^2})$, which is equivalent with $\frac{y^4}{y^2-y-a} \leq y^2+y+\frac{9}{5}$ and supposing $y^2-y-a>0$, it will be equivalent with $(\frac{4}{5}-a)y^2-(a+\frac{9}{5})y-\frac{9a}{5} \geq 0$ and this force $\frac{4}{5}-a>0$, or $a<\frac{4}{5}$. Let $a=\frac{4}{5}-\epsilon$ for some $\epsilon>0$. Therefore we should study $\frac{1}{y-1-\frac{\frac{4}{5}-\epsilon}{y}} \leq \frac{1}{y}(1+\frac{1}{y}+\frac{9}{5y^2})$, which is equivalent with

$$\frac{25\epsilon y^2+(25\epsilon-65)y+(45\epsilon-36)}{5y^3\big(5y^2-5y+(5\epsilon-4)\big)} \geq 0. \tag{2.1}$$

The equation $25\epsilon y^2+(25\epsilon-65)y+(45\epsilon-36)=0$ has discriminant $\Delta_1 = 169+14\epsilon-155\epsilon^2$, which is non-negative for $-1 \leq \epsilon \leq \frac{169}{155}$ and the greater root of it, is $y_1 = \frac{13-5\epsilon+\sqrt{\Delta_1}}{10\epsilon}$. Also, the equation $5y^2-5y+(5\epsilon-4)=0$ has discriminant $\Delta_2 = 105-100\epsilon$, which is non-negative for $\epsilon \leq \frac{21}{20}$ and the greater root of it, is $y_2 = \frac{1}{2}+\frac{\sqrt{\Delta_2}}{10}$. Thus, (2.1) holds for every $0<\epsilon \leq \min\{\frac{169}{155},\frac{21}{20}\} = \frac{21}{20}$, with $y \geq \max_{0<\epsilon\leq\frac{21}{20}}\{y_1,y_2\} = y_1$. Therefore, we have proved the following theorem:

Theorem 2.1. *For every $0<\epsilon\leq\frac{21}{20}$, the inequality*

$$\frac{x}{\log x - 1 - \frac{\frac{4}{5}-\epsilon}{\log x}} \leq \pi(x),$$

holds for all

$$x \geq \max\left\{32299, e^{\frac{13-5\epsilon+\sqrt{169+14\epsilon-155\epsilon^2}}{10\epsilon}}\right\}.$$

Corollary 2.2. *For every $x \geq 3299$, we have*

$$\frac{x}{\log x - 1 + \frac{1}{4\log x}} \leq \pi(x).$$

Proof. Taking $\epsilon = \frac{21}{20}$ in above theorem, we yield the result for $x \geq 32299$. For $3299 \leq x \leq 32298$ check it by computer. □

Upper Bounds. Similar to lower bounds, we should search suitable values of b in which $\pi(x) \leq \frac{x}{\log x-1-\frac{b}{\log x}}$. Considering (1.5) and letting $y=\log x$, we should study $\frac{1}{y}(1+\frac{1}{y}+\frac{251}{100y^2}) \leq \frac{1}{y-1-\frac{b}{y}}$. Assuming $y^2-y-b>0$, it will be equivalent with $(\frac{151}{100}-b)y^2-(b+\frac{251}{100})y-\frac{251b}{100} \leq 0$, which force $b \geq \frac{151}{100}$. Let $b=\frac{151}{100}+\epsilon$ for some $\epsilon \geq 0$. Therefore we should study $\frac{1}{y}(1+\frac{1}{y}+\frac{251}{100y^2}) \leq \frac{1}{y-1-\frac{\frac{151}{100}+\epsilon}{y}}$, which is equivalent with

$$\frac{10000\epsilon y^2+(10000\epsilon+40200)y+(25100\epsilon+37901)}{100y^3\big(100y^2-100y-(100\epsilon+151)\big)} \geq 0. \tag{2.2}$$

The quadratic equation in numerator of (2.2) has discriminant $\Delta_1 = 40401 - 17801\epsilon - 22600\epsilon^2$, which is non-negative for $-\frac{40401}{22600} \leq \epsilon \leq 1$ and the greater root of it, is $y_1 = \frac{-201-50\epsilon+\sqrt{\Delta_1}}{100\epsilon}$. Also, the quadratic equation in denominator of it, has discriminant $\Delta_2 = 44 + 25\epsilon$, which is non-negative for $-\frac{44}{25} \leq \epsilon$ and the greater root of it, is $y_2 = \frac{1}{2} + \frac{\sqrt{\Delta_2}}{5}$. Thus, (2.2) holds for every $0 \leq \epsilon \leq \min\{1, +\infty\} = 1$, with $y \geq \max_{0\leq\epsilon\leq 1}\{y_1, y_2\} = y_2$. Finally, we note that for $0 \leq \epsilon \leq 1$, $y_2(\epsilon)$ is strictly increasing and so, $6 < e^{\frac{1}{2}+\frac{\sqrt{44}}{5}} = e^{y_1(0)} \leq e^{y_1(\epsilon)} \leq e^{y_1(1)} = e^{\frac{1}{2}+\frac{\sqrt{69}}{5}} < 9$. Therefore, we obtain the following theorem:

Theorem 2.3. *For every $0 \leq \epsilon \leq 1$, we have*

$$\pi(x) \leq \frac{x}{\log x - 1 - \frac{\frac{151}{100}+\epsilon}{\log x}} \qquad (x \geq 355991).$$

Corollary 2.4. *For every $x \geq 7$, we have*

$$\pi(x) \leq \frac{x}{\log x - 1 - \frac{151}{100\log x}}.$$

Proof. Taking $\epsilon = 0$ in above theorem, we yield the result for $x \geq 355991$. For $7 \leq x \leq 35991$ it has been checked by computer [4]. □

3. Bounds of the form $\frac{n}{H_n - c}$ and $\frac{x}{\Psi(x)-c}$

Theorem 3.1. *(i) For every $n \geq 3299$, we have*

$$\frac{n}{H_n - a} < \pi(n),$$

in which $a = \gamma + 1 - \frac{1}{4\log 3299} \approx 1.546356705$.
(ii) For every $n \geq 9$, we have

$$\pi(n) < \frac{n}{H_n - b},$$

in which $b = 2 + \frac{151}{100\log 7} \approx 2.775986497$.

Proof. For $n \geq 3299$, we have $\gamma + \log n > a + \log n - 1 + \frac{1}{4\log n}$ and considering this with left hand side of (1.1), we obtain $\frac{n}{H_n - a} < \frac{x}{\log x - 1 + \frac{1}{4\log x}}$ and this inequality with corollary 2.2, yield the first part of theorem.
For $n \geq 9$, we have $b + \log n - 1 - \frac{151}{100\log n} \geq 1 + \log n$ and considering this with right hand side of (1.1), we obtain $\frac{x}{\log x - 1 - \frac{151}{100\log x}} < \frac{n}{H_n - b}$. Considering this, with corollary 2.4, complete the proof. □

Theorem 3.2. *(i) For every $x \geq 3299$, we have*

$$\frac{x}{\Psi(x) - A} < \pi(x),$$

in which $A = \frac{3298}{3299} - \frac{1}{4\log 3299} \approx 0.9688379174$.
(ii) For every $x \geq 9$, we have

$$\pi(x) < \frac{x}{\Psi(x) - B},$$

in which $B = 2 + \frac{151}{100\log 7} - \gamma \approx 2.198770832$.

Proof. Let H_x be the step function defined by $H_x = H_n$ for $n \leq x < n+1$. Considering (1.2) we have $H(x-1) < H_x \leq H(x)$.
For $x \geq 3299$ by considering part (i) of previous theorem, we have

$$\pi(x) > \frac{x}{H_x - a} \geq \frac{x}{H(x)-a} = \frac{x}{\Psi(x+1)+\gamma-a}.$$

Thus, by considering (1.3) we obtain

$$\pi(x) > \frac{x}{\Psi(x)+\frac{1}{x}+\gamma-a} \geq \frac{x}{\Psi(x)+\frac{1}{3299}+\gamma-a} = \frac{x}{\Psi(x)-A},$$

in which $A = a-\gamma-\frac{1}{3299} = \frac{3298}{3299} - \frac{1}{4\log 3299}$.
For $x \geq 9$ by considering second part of previous theorem, we obtain

$$\pi(x) < \frac{x}{H_x - b} < \frac{x}{H(x-1)-b} = \frac{x}{\Psi(x)+\gamma-b} = \frac{x}{\Psi(x)-B},$$

in which $B = b-\gamma = 2+\frac{151}{100\log 7}-\gamma$, and this completes the proof. □

Inverse of this theorem seems to be nice; because using it, for every $x \geq 3299$ we obtain

$$\frac{x}{\pi(x)} + A < \Psi(x) < \frac{x}{\pi(x)} + B.$$

Moreover, considering this inequality with (1.4) and (1.5), we yield the following bounds for $x \geq 355991$

$$\frac{\log x}{1+\frac{1}{\log x}+\frac{2.51}{\log^2 x}} + A < \Psi(x) < \frac{\log x}{1+\frac{1}{\log x}+\frac{1.8}{\log^2 x}} + B.$$

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