A REFINEMENT OF MANDL'S INEQUALITY

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Abstract. In this short note, we prove that $\frac{n}{2}p_n - \sum_{i=1}^{n} p_i > 0.01659n^2$ holds for every $n \geq 9$. This is a refinemet of Mandl's inequality which asserts $\frac{n}{2}p_n - \sum_{i=1}^{n} p_i > 0$, for those values of n.

1. Introduction and Refinement

As usual, let p_n be the n^{th} prime. The Mondl's conjecture (see [1] and [2]) asserts that for every $n \geq 9$, we have:

$$\frac{n}{2}p_n - \sum_{i=1}^{n} p_i > 0.$$

To prove Mandl's inequality, Dusart ([1], page 50) uses the following inequality

$$\int_2^{p_n} \pi(t)dt \geq c + \frac{p_n^2}{2\log p_n}\left(1 + \frac{3}{2\log p_n}\right) \qquad (n \geq 109), \tag{1.1}$$

in which

$$c = 35995 - 3Li(599^2) + \frac{599^2}{\log 599} \approx -47.06746,$$

and

$$Li(x) = \lim_{\epsilon \to 0}\left(\int_0^{1-\epsilon} \frac{dt}{\log t} + \int_{1+\epsilon}^{x} \frac{dt}{\log t}\right),$$

is logarithmic function, and base of all logarithms are e. Note that one can gets (1.1), using the following known bound [1]:

$$\pi(t) \geq \frac{t}{\log t}\left(1 + \frac{1}{\log t}\right) \qquad (t \geq 599).$$

Also, for using (1.1) to prove Mandl's inequality, we note that:

$$\int_2^{p_n} \pi(t)dt = \sum_{i=2}^{n} \big(p_i - p_{i-1}\big)(i-1) = \sum_{i=2}^{n} \big(ip_i - (i-1)p_{i-1}\big) - \sum_{i=2}^{n} p_i = np_n - \sum_{i=1}^{n} p_i.$$

Therefore, we have:

$$np_n - \sum_{i=1}^{n} p_i \geq c + \frac{p_n^2}{2\log p_n}\left(1 + \frac{3}{2\log p_n}\right) \qquad (n \geq 109). \tag{1.2}$$

Now, we use the following bound ([1], page 36):

$$\frac{x}{\log x}\left(1 + \frac{1.2762}{\log x}\right) \geq \pi(x) \qquad (x \geq 2).$$

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Considering this bound and (1.2), for every $n \geq 109$, we obtain:

$$np_n - \sum_{i=1}^{n} p_i \geq c + \frac{p_n^2}{2\log p_n}\left(\frac{0.2238}{\log p_n}\right) + \frac{p_n^2}{2\log p_n}\left(1 + \frac{1.2762}{\log p_n}\right)$$
$$\geq c + 0.1119\frac{p_n^2}{\log^2 p_n} + \frac{p_n}{2}\pi(p_n) = c + 0.1119\frac{p_n^2}{\log^2 p_n} + \frac{n}{2}p_n.$$

So, for every $n \geq 109$, we have:

$$\frac{n}{2}p_n - \sum_{i=1}^{n} p_i \geq c + 0.1119\frac{p_n^2}{\log^2 p_n}. \tag{1.3}$$

In other hand, we have the following bounds for p_n ([3], page 69):

$$n\log n \leq p_n \leq n(\log n + \log\log n) \qquad (n \geq 6).$$

Combining these bounds with (1.3), for every $n \geq 109$, we yield that:

$$\frac{n}{2}p_n - \sum_{i=1}^{n} p_i \geq c + \frac{0.1119(n\log n)^2}{\log^2\big(n(\log n + \log\log n)\big)}.$$

But, for every $n \geq 89$, we have $c + \frac{0.0119(n\log n)^2}{\log^2(n(\log n+\log\log n))} > 0$, and so, we obtain the following inequality for every $n \geq 89$:

$$\frac{n}{2}p_n - \sum_{i=1}^{n} p_i \geq \frac{(n\log n)^2}{10\log^2\big(n(\log n + \log\log n)\big)}.$$

This holds also for $10 \leq n \leq 88$. Thus, considering $\log(n(\log n + \log\log n)) < 2\log n + 1$, we yield that:

$$\frac{n}{2}p_n - \sum_{i=1}^{n} p_i \geq \frac{n^2}{10}\left(\frac{\log n}{2\log n + 1}\right)^2,$$

which holds for every $n \geq 9$. This is a refinement of Mandl's ineqiality, with quadratic and logarithms terms. Now, for every $n \geq 9$, we note that:

$$\frac{n}{2}p_n - \sum_{i=1}^{n} p_i \geq \frac{n^2}{10}\left(\frac{\log n}{2\log n + 1}\right)^2 \geq \frac{n^2}{10}\left(\frac{\log 9}{2\log 9 + 1}\right)^2 > 0.01659n^2.$$

References

[1] Pierre DUSART, Autour de la fonction qui compte le nombre de nombres premiers, PhD. Thesis, 1998.

[2] J. Barkley Rosser & L. Schoenfeld, Sharper Bounds for the Chebyshev Functions $\theta(x)$ and $\psi(x)$, *Math. Of Computation*, Vol. 29, Number 129 (January 1975) pp. 243-269.

[3] J. Barkley Rosser & L. Schoenfeld, Approximate Formulas for Some Functions of Prime Numbers, *Illinois Journal Math.*, 6 (1962) pp. 64-94.

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