ON THE HOMOGENEOUS FUNCTIONS WITH TWO PARAMETERS AND ITS MONOTONICITY

ZHEN-HANG YANG

Abstract. Suppose $f(x,y)$ is a positive homogeneous function defined on $\mathbb{U}(\subseteqq \mathbb{R}_+ \times \mathbb{R}_+)$, call $H_f(a,b;p,q) = \left[\frac{f(a^p,b^p)}{f(a^q,b^q)}\right]^{\frac{1}{p-q}}$ homogeneous function with two parameters. If $f(x,y)$ is 2nd differentiable, then the monotonicity for parameters p and q of $H_f(a,b;p,q)$ depend on the sign of $I_1 = (\ln f)_{xy}$, for variable a and b depend on the sign of $I_{2a} = [(\ln f)_x \ln(y/x)]_y$ and $I_{2b} = [(\ln f)_y \ln(x/y)]_x$ respectively. As applications of these results, a serial of inequalities for arithmetic mean, geometric mean, exponential mean, logarithmic mean, power-Exponential mean and exponential-geometric mean are deduced.

1. Introduction

The so-called two-parameter mean or extended mean values between two unequal positive numbers a and b were defined first by K.B. Stolarsky in [17] as

$$E(a,b;p,q) = \begin{cases} \left(\dfrac{q(a^p-b^p)}{p(a^q-b^q)}\right)^{\frac{1}{p-q}} & p \neq q, pq \neq 0 \\ \left(\dfrac{a^p-b^p}{p(\ln a-\ln b)}\right)^{\frac{1}{p}} & p \neq 0, q = 0 \\ \left(\dfrac{a^q-b^q}{q(\ln a-\ln b)}\right)^{\frac{1}{q}} & p = 0, q \neq 0 \\ \exp\left(\dfrac{a^p\ln a-b^p\ln b}{a^p-b^p}-\dfrac{1}{p}\right) & p = q \neq 0 \\ \sqrt{ab} & p = q = 0 \end{cases}. \tag{1.1}$$

Date: September. 10, 2004.

2000 *Mathematics Subject Classification.* Primary 26B35, 26E60; Secondary 26A48, 26D07.

Key words and phrases. homogeneous function with two parameters, f-mean with two-parameter, monotonicity, estimate for lower and upper bounds.

Thanks for Mr. Zhang Zhihua.

This paper is in final form and no version of it will be submitted for publication elsewhere.

The monotonicity of $E(a,b;p,q)$ has been researched by E. B. Leach and M. C. Sholander in [4], and others also in [5–9, 11, 14, 15, 17] using different ideas and simpler methods.

As the generalized power-mean, C. Gini obtained a similar two-parameter type mean in [1]. That is:

$$G(a,b;p,q)=\begin{cases}\left(\dfrac{a^p+b^p}{a^q+b^q}\right)^{\frac{1}{p-q}} & p\neq q\\ \exp\left(\dfrac{a^p\ln a+b^p\ln b}{a^p+b^p}\right) & p=q\neq 0\\ \sqrt{ab} & p=q=0\end{cases}. \tag{1.2}$$

Recently, the sufficient and necessary conditions comparing two-parameter mean with Gini mean were put forward by using the so-called concept of “strong inequalities”([3]).

From the above two-parameter type means, we find that their forms are both $\left(\dfrac{f(a^p,b^p)}{f(a^q,b^q)}\right)^{\frac{1}{p-q}}$, and where $f(x,y)$ is a homogeneous function for x and y.

The main aim of this paper is to establish the concept as “two-parameter homogeneous function”, and study the monotonicity of function in the form of $\left(\dfrac{f(a^p,b^p)}{f(a^q,b^q)}\right)^{\frac{1}{p-q}}$. As applications for main results, we will deduce three inequality’s chains which contain arithmetic mean, geometric mean, exponential mean, logarithmic mean, power-exponential mean and exponential-geometric mean, prove an upper bound for Stolarsky mean in [12], present two estimated expressions for exponential mean.

2. Basic concepts and main results

Definition 2.1. *Assume f: $\mathbb{U}(\subseteq \mathbb{R}_+\times\mathbb{R}_+)\to\mathbb{R}_+$ is a homogeneous function for variable x and y, and is continual and exist 1st partial derivative, $(a,b)\in\mathbb{R}_+\times\mathbb{R}_+$ and $a\neq b$, $(p,q)\in\mathbb{R}\times\mathbb{R}$. If $(1,1)\notin\mathbb{U}$, then Define that*

$$\mathcal{H}_f(a,b;p,q) = \left[\frac{f(a^p,b^p)}{f(a^q,b^q)}\right]^{\frac{1}{p-q}}\ (p\neq q, pq\neq 0), \tag{2.1}$$

$$\mathcal{H}_f(a,b;p,p) = \lim_{q\to p}\mathcal{H}_f(a,b;p,q)=G_{f,p}(a,b)(p=q\neq 0). \tag{2.2}$$

where $G_{f,p}(a,b)=G_f^{\frac{1}{p}}(a^p,b^p)$, $G_f(x,y)=\exp\left[\dfrac{xf_x(x,y)\ln x+yf_y(x,y)\ln y}{f(x,y)}\right]$, $f_x(x,y)$ or $f_y(x,y)$ is to calculate partial derivative to 1st or 2nd variable of $f(x,y)$ respectively.

If $(1,1) \in \mathbb{U}$, then define further

$$(2.3)\quad \mathcal{H}_f(a,b;p,0) = \left[\frac{f(a^p,b^p)}{f(1,1)}\right]^{\frac{1}{p}} (p \neq 0, q=0),$$

$$(2.4)\quad \mathcal{H}_f(a,b;0,q) = \left[\frac{f(a^q,b^q)}{f(1,1)}\right]^{\frac{1}{q}} (p=0, q \neq 0),$$

$$(2.5)\quad \mathcal{H}_f(a,b;0,0) = \lim_{p\to 0}\mathcal{H}_f(a,b;p,0) = a^{\frac{f_x(1,1)}{f(1,1)}} b^{\frac{f_y(1,1)}{f(1,1)}} (p=q=0).$$

From Lemma 3.1, $\mathcal{H}_f(a,b;p,q)$ is still a homogeneous function for positive numbers a and b, we call it a homogeneous function for positive numbers a and b with two parameters p and q, simply call it two-parameter homogeneous function. In the case of not being confused, we also denote it $\mathcal{H}_f(p,q)$ or $\mathcal{H}_f(a,b)$ or $\mathcal{H}_f$.

If $f(x,y)$ is a positive 1-order homogeneous mean function defined on $\mathbb{R}_+ \times \mathbb{R}_+$, then call $\mathcal{H}_f(a,b;p,q)$ two-parameter f-mean for positive numbers a and b .

Remark 2.1. *If $f(x,y)$ is a positive 1-order homogeneous function defined on $\mathbb{R}_+ \times \mathbb{R}_+$, and is continual and exist 1st partial derivative, and satisfies $f(x,y) = f(y,x)$, then $G_{f,0}(a,b) = \mathcal{H}_f(a,b;0,0) = \sqrt{ab}$.*

In fact,

$$G_f{}^{\frac{1}{p}}(a^p,b^p) = \exp\left[\frac{a^p f_x(a^p,b^p)\ln a + y^p f_y(a^p,b^p)\ln b}{f(a^p,b^p)}\right],$$

so

$$G_{f,0}(a,b) = \exp\left[\frac{f_x(1,1)\ln a + f_y(1,1)\ln b}{f(1,1)}\right] = \mathcal{H}_f(a,b;0,0).$$

Since $f(x,y)$ is a positive 1-order homogeneous function, from expression (3.1) of Lemma 3.2 , we obtain

$$(2.6)\qquad \frac{1\cdot f_x(1,1)}{f(1,1)} + \frac{1\cdot f_y(1,1)}{f(1,1)} = 1.$$

If $f(x,y) = f(y,x)$, then $f_x(x,y) = f_y(y,x)$, we have

$$(2.7)\qquad f_x(1,1) = f_y(1,1).$$

By (2.6) and (2.7), we get $\dfrac{f_x(1,1)}{f(1,1)} = \dfrac{f_y(1,1)}{f(1,1)} = \dfrac{1}{2}$, thereby $G_{f,0} = \sqrt{ab}$.

Thus it can be seen that despite the form of $f(x,y)$ we have always $\mathcal{H}_f(a,b;0,0) = G_{f,0}(a,b) = \sqrt{ab}$, so long as $f(x,y)$ is a positive 1-order homogeneous symmetric function defined on $\mathbb{R}_+ \times \mathbb{R}_+$.

Example 2.1. *In Definition 2.1, let* $f(x,y) = L(x,y) = \dfrac{x-y}{\ln x - \ln y}$ *(*$x, y > 0, x \neq y$*, as the same below), we get (1.1), i.e.*

$$\mathcal{H}_L(a,b;p,q) = \begin{cases} \left(\dfrac{q(a^p-b^p)}{p(a^q-b^q)}\right)^{\frac{1}{p-q}} & p \neq q, pq \neq 0 \\ L^{\frac{1}{p}}(a^p,b^p) & p \neq 0, q = 0 \\ L^{\frac{1}{q}}(a^q,b^q) & p = 0, q \neq 0 \\ G_{L,p}(a,b) & p = q \neq 0 \\ G(a,b) & p = q = 0 \end{cases}, \tag{2.8}$$

where $G_{L,p}(a,b) = E_p(a,b) = E^{\frac{1}{p}}(a^p,b^p) = E_p, E(a,b) = e^{-1}\left(\dfrac{a^a}{b^b}\right)^{\frac{1}{a-b}}$*,* $G(a,b) = \sqrt{ab}$*.*

Example 2.2. *In Definition 2.1, let* $f(x,y) = A(x,y) = \dfrac{x+y}{2}$ *(*$x, y > 0, x \neq y$*, as the same below), we get (1.2), i.e.*

$$\mathcal{H}_A(a,b;p,q) = \begin{cases} \left(\dfrac{a^p+b^p}{a^q+b^q}\right)^{\frac{1}{p-q}} & p \neq q \\ G_{A,p}(a,b) & p = q \neq 0 \\ G(a,b) & p = q = 0 \end{cases}, \tag{2.9}$$

where $G_{A,p}(a,b) = Z_p(a,b) = Z^{\frac{1}{p}}(a^p,b^p) = Z_p.Z(a,b) = a^{\frac{a}{a+b}}b^{\frac{b}{a+b}}$ *is named power-exponential mean between positive numbers a and b.*

Example 2.3. *In Definition 2.1, let* $f(x,y) = E(x,y) = e^{-1}\left(\dfrac{x^x}{y^y}\right)^{\frac{1}{x-y}}$ *(*x*,* $y > 0$*,* $x \neq y$*, as the same below), then*

$$\mathcal{H}_E(a,b;p,q) = \begin{cases} \left(\dfrac{E(a^p,b^p)}{E(a^q,b^q)}\right)^{\frac{1}{p-q}} & p \neq q \\ G_{E,p}(a,b) & p = q \neq 0 \\ G(a,b) & p = q = 0 \end{cases}, \tag{2.10}$$

where $G_{E,p}(a,b) = Y_p(a,b) = Y^{\frac{1}{p}}(a^p,b^p) = Y_p$*.* $Y(a,b) = Ee^{1-\frac{G^2}{L^2}}$ *is named exponential-geometric mean between positive numbers a and b, where* $E = E(a,b), L = L(a,b), G = G(a,b)$*.*

Example 2.4. *In Definition 2.1, let* $f(x,y) = D(x,y) = |x-y|$ *(*$x, y > 0, x \neq y$*, as the same below), then*

$$\mathcal{H}_D(a,b;p,q) = \begin{cases} \left|\dfrac{a^p-b^p}{a^q-b^q}\right|^{\frac{1}{p-q}} & p \neq q, pq \neq 0 \\ G_{D,p}(a,b) & p = q \neq 0 \end{cases}, \tag{2.11}$$

where $G_{D,p}(a,b) = G_{D,p} = e^{\frac{1}{p}}E^{\frac{1}{p}}(a^p,b^p) = e^{\frac{1}{p}}E_p$*.*

In order to avoid confusion, we rename $\mathcal{H}_L(a,b;p,q)$ (or $E(a,b;p,q)$) and $\mathcal{H}_A(a,b;p,q)$ (or $G(a,b;p,q)$) as two-parameter logarithmic mean and two-parameter arithmetic mean respectively. In the same way, we call $\mathcal{H}_E(a,b;p,q)$ in Example 2.3 two-parameter exponential mean.

In Example 2.4, since $D(x,y) = |x-y|$ is not a certain mean between positive numbers x and y, but one absolute value function of difference of two positive numbers, we call $\mathcal{H}_D(a,b;p,q)$ two-parameter homogeneous function of difference.

It is obvious that the conception of two-parameter homogeneous functions have developed greatly the extension of conception of two-parameter mean.

For monotonicity of two-parameter homogeneous function $\mathcal{H}_f(a,b;p,q)$, we have the following main results.

Theorem 2.1. *Let $f(x,y)$ be a positive n-order homogenous function defined on $\mathbb{U}(\subseteqq \mathbb{R}_+ \times \mathbb{R}_+)$, and be 2nd differentiable. If $I_1 = (\ln f)_{xy} \underset{(<)}{>} 0$, then $\mathcal{H}_f(p,q)$ is strictly monotone increasing (decreasing) for p or q respectively.*

Corollary 2.1. *1) when $p,q \in (-\infty,+\infty)$, $\mathcal{H}_L(p,q), \mathcal{H}_A(p,q), \mathcal{H}_E(p,q)$ is strictly monotone increasing for p or q respectively.*

2) when $p,q \in (-\infty,0) \cup (0,+\infty)$, $\mathcal{H}_D(p,q)$ is strictly monotone decreasing for p or q respectively.

Theorem 2.2. *Let $f(x,y)$ be a positive 1-order homogeneous function defined on $\mathbb{U}(\subseteqq \mathbb{R}_+ \times \mathbb{R}_+)$, and be 2nd differentiable.*

1) If $I_{2a} = [(\ln f)_x \ln(y/x)]_y \underset{(<0)}{>} 0$, then $\mathcal{H}_f(a,b)$ is strictly monotone increasing (decreasing) for a.

2) If $I_{2b} = [(\ln f)_y \ln(x/y)]_x \underset{(<)}{>} 0$, then $\mathcal{H}_f(a,b)$ is strictly monotone increasing (decreasing) for b.

Corollary 2.2. *$\mathcal{H}_L(a,b), \mathcal{H}_D(a,b)$ is strictly monotone increasing for a or b respectively.*

3. Lemmas and proofs of the main results

In order to prove the main results in this article, we need some properties for homogeneous function in [16]. For convenience, we quote them as follows.

Lemma 3.1. *Let $f(x,y)$, $g(x,y)$ be an n,m-order homogenous function over Ω respectively, then $f \cdot g$, $f/g(g \neq 0)$ is an $n+m, n-m$-order homogenous function over Ω respectively.*

If for a certain p and $(x^p,y^p) \in \Omega$, $f^p(x,y)$ exist, then $f(x^p,y^p)$, $f^p(x,y)$ are both np-order homogeneous functions over Ω.

Lemma 3.2. *Let $f(x,y)$ be a n-order homogeneous function over Ω, and f_x, f_y both exist, then f_x, f_y are both $(n-1)$-order homogeneous functions over Ω, furthermore we have*

$$xf_x + yf_y = nf. \tag{3.1}$$

In particular, when $n = 1$ and $f(x,y)$ is 1st differentiable over Ω, then

$$\begin{aligned} xf_x + yf_y &= f, && (3.2)\\ xf_{xx} + yf_{xy} &= 0, && (3.3)\\ xf_{xy} + yf_{yy} &= 0. && (3.4) \end{aligned}$$

Lemma 3.3. *Let $f(x,y)$ be a positive n-order homogenous function defined on $\mathbb{U}(\subseteqq \mathbb{R}_+ \times \mathbb{R}_+)$, and be 2nd differentiable. Denote $T(t) = \ln f(a^t, b^t), x = a^t, y = b^t, a, b > 0$, then $T''(t) = -xyI_1(\ln b - \ln a)^2$, where*

$$I_1 = \frac{\partial^2 \ln f(x,y)}{\partial x \partial y} = (\ln f(x,y))_{xy} = (\ln f)_{xy}.$$

Proof. Since $f(x,y)$ is a positive n-order homogeneous function, from expression (3.1), we can obtain $x(\ln f)_x + y(\ln f)_y = n$ or $x(\ln f)_x = n - y(\ln f)_y$, $y(\ln f)_y = n - x(\ln f)_x$,

$$\begin{aligned} T'(t) &= \frac{a^t f_x(a^t, b^t)\ln a + b^t f_y(a^t, b^t)\ln b}{f(a^t, b^t)} && (3.5)\\ &= \frac{xf_x(x,y)\ln a + yf_y(x,y)\ln b}{f(x,y)} && (3.6)\\ &= x(\ln f)_x \ln a + y(\ln f)_y \ln b. && (3.7) \end{aligned}$$

Hence

$$\begin{aligned} T''(t) &= \frac{\partial T'(t)}{\partial x}\frac{dx}{dt} + \frac{\partial T'(t)}{\partial y}\frac{dy}{dt}\\ &= \left[y(\ln f)_y(\ln b - \ln a) + n\ln a\right]_x a^t \ln a +\\ &\quad \left[x(\ln f)_x(\ln a - \ln b) + n\ln b\right]_y b^t \ln b\\ &= y(\ln f)_{yx}(\ln b - \ln a)x\ln a + x(\ln f)_{xy}(\ln a - \ln b)y\ln b\\ &= -xy(\ln f)_{xy}(\ln b - \ln a)^2 = -xyI_1(\ln b - \ln a)^2. \end{aligned}$$

∎

Lemma 3.4. *Let $f(x,y)$ be a positive 1-order homogeneous function defined on $\mathbb{U}(\subseteqq \mathbb{R}_+ \times \mathbb{R}_+)$, and be 2nd differentiable, denote $S(t) = \dfrac{txf_x(x,y)}{f(x,y)}, x = a^t, y = b^t\ a, b > 0$, then $S'(t) = xyI_{2a}$, where*

$$I_{2a} = \left[(\ln f)_x \ln(y/x)\right]_y.$$

Proof.

$$\begin{aligned}
S'(t) &= \frac{xf_x(x,y)}{f(x,y)} + t\frac{d}{dt}\left[\frac{xf_x(x,y)}{f(x,y)}\right] \\
&= x(\ln f)_x + t\left[\frac{\partial(x(\ln f)_x)}{\partial x}\frac{dx}{dt} + \frac{\partial(x(\ln f)_x)}{\partial y}\frac{dy}{dt}\right] \\
&= x(\ln f)_x + t\left[\frac{\partial(x(\ln f)_x)}{\partial x}a^t\ln a + \frac{\partial(x(\ln f)_x)}{\partial y}b^t\ln b\right] \\
&= x(\ln f)_x + t\left[x(x(\ln f)_x)_x\ln a + y(x(\ln f)_x)_y\ln b\right].
\end{aligned}$$

By Lemma 3.1, that $x(\ln f)_x = \dfrac{xf_x(x,y)}{f(x,y)}$ is a 0-order homogeneous function, from expression (3.1) of Lemma 3.2, we obtain $x\left[x(\ln f)_x\right]_x + y\left[x(\ln f)_x\right]_y = 0$ or $x\left[x(\ln f)_x\right]_x = -y\left[x(\ln f)_x\right]_y$,

hence

$$\begin{aligned}
S'(t) &= x(\ln f)_x + ty\left[x(\ln f)_x\right]_y(\ln b - \ln a) \\
&= x(\ln f)_x + txy(\ln f)_{xy}(\ln b - \ln a) \\
&= x(\ln f)_x + xy(\ln f)_{xy}(\ln b^t - \ln a^t) \\
&= x(\ln f)_x + xy(\ln f)_{xy}(\ln y - \ln x) \\
&= xy\left[y^{-1}(\ln f)_x + (\ln f)_{xy}\ln(y/x)\right] \\
&= xy\left[(\ln f)_x\ln(y/x)\right]_y = xyI_{2a}.
\end{aligned}$$

∎

Based on the above Lemmas, then next we will go on proving main results in this paper.

proof of theorem 2.1. Since $\mathcal{H}_f(p,q)$ is symmetric for p and q, it needs only to prove the monotonicity for p of $\ln\mathcal{H}_f$.

1) when $p \neq q$,

$$\begin{aligned}
\ln\mathcal{H}_f &= \frac{1}{p-q}\ln\frac{f(a^p,b^p)}{f(a^q,b^q)} = \frac{T(p)-T(q)}{p-q}, \\
\frac{\partial\ln\mathcal{H}_f}{\partial p} &= \frac{(p-q)T'(p)-T(p)+T(q)}{(p-q)^2}.
\end{aligned}$$

Denote $g(p) = (p-q)T'(p) - T(p) + T(q)$, then $g(q) = 0$, $g'(p) = (p-q)T''(p)$, and then exist $\xi = q + \theta(p-q), \theta \in (0,1)$ by mean-value theorem, such that

$$\frac{\partial\ln\mathcal{H}_f}{\partial p} = \frac{g(p)-g(q)}{(p-q)^2} = \frac{g'(\xi)}{p-q} = \frac{(\xi-q)T''(\xi)}{p-q} = (1-\theta)T''(\xi).$$

By Lemma 3.3, $T''(\xi) = -xyI_1(\ln b - \ln a)^2$, $x = a^\xi, y = b^\xi$. Obviously, when $I_1 \underset{(>)}{<} 0$, we get $\dfrac{\partial\ln\mathcal{H}_f}{\partial p} \underset{(<)}{>} 0$.

2) when $p=q$, from (2.2) and (3.6),

$$\ln \mathcal{H}_f = \ln G_f^{\frac{1}{p}}(a^p,b^p) = \frac{xf_x(x,y)\ln x + yf_y(x,y)\ln y}{f(x,y)} = T'(p),$$

$$\frac{\partial \ln \mathcal{H}_f}{\partial p} = T''(p) = -xyI_1(\ln b - \ln a)^2.$$

when $I_1 \underset{(>)}{<} 0$, we get $\frac{\partial \ln \mathcal{H}_f}{\partial p} \underset{(<)}{>} 0$.

Combining 1) with 2), we draw a conclusion of this theorem immediately. ∎

proof of corollary 2.1. It follows that theorem 2.1, the monotonicity of $\mathcal{H}_f(p,q)$ depends on the sign of $I_1 = (\ln f)_{xy}$.

1) When $f(x,y)=L(x,y)$,

$$\begin{aligned} I_1 &= (\ln f)_{xy} = \frac{1}{(x-y)^2} - \frac{1}{xy(\ln x - \ln y)^2} \\ &= \frac{1}{xy(x-y)^2}\left((\sqrt{xy})^2 - L^2(x,y)\right). \end{aligned}$$

By well-known inequality $L(x,y) > \sqrt{xy}$([13]), $I_1 < 0$.

2) When $f(x,y)=A(x,y)$,

$$I_1 = (\ln f)_{xy} = -\frac{1}{(x+y)^2} < 0.$$

3) When $f(x,y)=E(x,y)$,

$$\begin{aligned} I_1 &= (\ln f)_{xy} = \frac{1}{(x-y)^3}\left[2(x-y) - (x+y)(\ln x - \ln y)\right] \\ &= \frac{2(\ln x - \ln y)}{(x-y)^3}\left[L(x,y) - \frac{x+y}{2}\right]. \end{aligned}$$

By well-known inequality $L(x,y) < \frac{x+y}{2}$([17]), $I_1 < 0$.

4) When $f(x,y)=D(x,y)$,

$$I_1 = (\ln f)_{xy} = \frac{1}{(x-y)^2} > 0.$$

Applying mechanically Theorem 2.1, we immediately obtain Corollary 2.1. ∎

proof of theorem 2.2. 1) Because

$$\frac{\partial \ln \mathcal{H}_f}{\partial a} = \frac{1}{p-q}\left[\frac{pa^{p-1}f_x(a^p,b^p)}{f(a^p,b^p)} - \frac{qa^{q-1}f_x(a^q,b^q)}{f(a^q,b^q)}\right] = \frac{S(p)-S(q)}{a(p-q)},$$

by mean-value theorem, there exist $\xi = q+\theta(p-q), \theta \in (0,1)$, such that

$$\frac{\partial \ln \mathcal{H}_f}{\partial a} = \frac{S(p)-S(q)}{a(p-q)} = a^{-1}S'(\xi).$$

From Lemma 3.4, $S'(\xi) = xyI_{2a}$, $x = a^{\xi}, y = b^{\xi}$. Obviously ,if $I_{2a} > 0$, then $\frac{\partial \ln \mathcal{H}_f}{\partial a} > 0$, so $\mathcal{H}_f(a,b)$ is strictly monotone increasing for a; If $I_{2a} < 0$, then $\frac{\partial \ln \mathcal{H}_f}{\partial a} < 0$, so $\mathcal{H}_f(a,b)$ is strictly monotone decreasing for a.

2) It can be proved in the same way. ∎

proof of corollary 2.2. 1) When $f(x,y) = L(x,y)$,

$$I_{2a} = [(\ln f)_x \ln(y/x)]_y = \frac{x/y - 1 - \ln(x/y)}{(x-y)^2}.$$

By well-known inequality $\ln x < x - 1 (x > 0, x \neq 1), I_{2a} > 0$.

2) When $f(x,y) = D(x,y)$,

$$I_{2a} = [(\ln f)_x \ln(y/x)]_y = \frac{x/y - 1 - \ln(x/y)}{(x-y)^2} > 0.$$

Since $\mathcal{H}_L(a,b), \mathcal{H}_D(a,b)$ are symmetric for a and b, applying mechanically Theorem 2.2, we immediately obtain Corollary 2.2. ∎

4. Some applications

As direct applications of theorems and lemmas in this paper, we will present several examples as follows.

Example 4.1. ***a G-A inequality's chain.*** *By 1) of Corollary 2.1, when $f(x,y) = A(x,y), L(x,y)$ and $E(x,y)$, $\mathcal{H}_f(p,q)$ are strictly monotone increasing for p or q. So there is*

$$\begin{aligned} \mathcal{H}_f(a,b;0,0) &< \mathcal{H}_f(a,b;1,0) < \mathcal{H}_f(a,b;1,\frac{1}{2}) \\ &< \mathcal{H}_f(a,b;1,1) < \mathcal{H}_f(a,b;1,2). \end{aligned} \tag{4.1}$$

From it we can obtain the following inequalities respectively that

$$\sqrt{ab} < L(a,b) < \left(\frac{\sqrt{a}+\sqrt{b}}{2}\right)^2 < E(a,b) < \frac{a+b}{2}, \tag{4.2}$$

$$\sqrt{ab} < \frac{a+b}{2} < \left(\frac{a+b}{\sqrt{a}+\sqrt{b}}\right)^2 < Z(a,b) < \frac{a^2+b^2}{a+b}, \tag{4.3}$$

$$\sqrt{ab} < E(a,b) < \left[\frac{E(a,b)}{E\left(\sqrt{a},\sqrt{b}\right)}\right]^2 < Y(a,b) < \frac{E(a^2,b^2)}{E(a,b).} \tag{4.4}$$

Notice $\frac{E(a^2,b^2)}{E(a,b)} = Z(a,b)$, then (4.4) can be rewritten into that

$$\sqrt{ab} < E(a,b) < Z^2\left(\sqrt{a},\sqrt{b}\right) < E\exp(1 - \frac{G^2}{L^2}) < Z(a,b). \tag{4.5}$$

The inequality (4.2) was proved by [13], which show that can insert L, $\dfrac{A+G}{2}$ and E between G and A, so we call (4.2) G-A inequality's chain. (4.3) and (4.4) are the same in form completely, so we call (4.1) G-A inequality's chain for homogeneous function.

Remark 4.1. *That $\dfrac{E(a^2,b^2)}{E(a,b)} = Z(a,b)$ is a new identical equation for mean. In fact,*

$$\begin{aligned} E(a,b)Z(a,b) &= e^{-1}\left(\frac{b^b}{a^a}\right)^{\frac{1}{b-a}} b^{\frac{b}{b+a}} a^{\frac{a}{b+a}} = e^{-1} b^{\frac{b}{b+a}+\frac{b}{b-a}} a^{\frac{a}{b+a}-\frac{a}{b-a}} \\ &= e^{-1} b^{\frac{2b^2}{b^2-a^2}} a^{\frac{-2a^2}{b^2-a^2}} = e^{-1}\left(\frac{(b^2)^{b^2}}{(a^2)^{a^2}}\right)^{\frac{1}{b^2-a^2}} = E(a^2,b^2). \end{aligned}$$

It shows that $Z(a,b)$ is not only one "geometric mean", but also one ratio of one exponential mean to another. Thus inequalities involving $Z(a,b)$ may be translated into inequalities involving exponential mean.

Example 4.2. ***An estimate for upper bound of Stolarsky mean.*** *From 2) of Corollary 2.1, we can prove expediently an estimate for upper bound of Stolarsky mean presented by [12]: $S_p(a,b) < p^{\frac{1}{1-p}}(a+b)$ with $p > 2$, where $S_p(a,b) = \left(\dfrac{b^p-a^p}{p(b-a)}\right)^{\frac{1}{p-1}}$.*

In fact, from 2) of Corollary 2.1, when $p,q \in (-\infty,0)\cup(0,+\infty)$, $\mathcal{H}_D(p,q)$ is strictly monotone decreasing for p or q respectively, so when $p > 2$, we have $\mathcal{H}_D(a,b;1,p) < \mathcal{H}_D(a,b;1,2)$.

Notice

$$\mathcal{H}_D(a,b;p,1) = \left(\frac{a^p-b^p}{a-b}\right)^{\frac{1}{p-1}} = p^{\frac{1}{p-1}} S_p(a,b) \quad (p>0), \tag{4.6}$$

then when $p > 2$, we obtain $p^{\frac{1}{p-1}} S_p(a,b) < 2^{\frac{1}{2-1}} S_2(a,b) = a+b$, i.e. $S_p(a,b) < p^{\frac{1}{1-p}}(a+b)$.

Example 4.3. ***Reverse inequalities and estimate for exponential mean.*** *By 1) of Corollary 2.1, $\mathcal{H}_L(p,q)$ is strictly monotone increasing for p or q, so when $p_1 \in (0,1)$, $p_2 \in (1,+\infty)$, we have*

$$\mathcal{H}_L(a,b;p_1,1) < \mathcal{H}_L(a,b;1,1) < \mathcal{H}_L(a,b;p_2,1),$$

i.e.

$$S_{p_1}(a,b) < E(a,b) < S_{p_2}(a,b). \tag{4.7}$$

On the other hand, By 2) of Corollary 2.1, when $p,q \in (-\infty,0)\cup(0,+\infty)$, $\mathcal{H}_D(p,q)$ is strictly monotone decreasing for p or q respectively. So when $p_1 \in (0,1)$, $p_2 \in (1,+\infty)$, we have

$$\mathcal{H}_D(a,b;p_1,1) > \mathcal{H}_D(a,b;1,1) > \mathcal{H}_D(a,b;p_2,1). \tag{4.8}$$

From (4.6), (4.8) can be rewritten into

$$p_2^{\frac{1}{p_2-1}} S_{p_2}(a,b) < eE(a,b) < p_1^{\frac{1}{p_1-1}} S_{p_1}(a,b)$$

or

$$\frac{1}{e} p_2^{\frac{1}{p_2-1}} S_{p_2}(a,b) < E(a,b) < \frac{1}{e} p_1^{\frac{1}{p_1-1}} S_{p_1}(a,b). \tag{4.9}$$

Combining (4.7) with (4.9), we have

$$S_{p_1}(a,b) \;<\; E(a,b) < \frac{1}{e} p_1^{\frac{1}{p_1-1}} S_{p_1}(a,b), p_1 \in (0,1), \tag{4.10}$$

$$\frac{1}{e} p_2^{\frac{1}{p_2-1}} S_{p_2}(a,b) \;<\; E(a,b) < S_{p_2}(a,b), p_2 \in (1,+\infty). \tag{4.11}$$

In particular, when $p_1 = \dfrac{1}{2}, p_2 = 2$*, by (4.10), (4.11), we get*

$$\left(\frac{\sqrt{a}+\sqrt{b}}{2}\right)^2 \;<\; E(a,b) < \frac{4}{e}\left(\frac{\sqrt{a}+\sqrt{b}}{2}\right)^2, \tag{4.12}$$

$$\frac{2}{e}\left(\frac{a+b}{2}\right) \;<\; E(a,b) < \frac{a+b}{2}. \tag{4.13}$$

The inequalities (4.12) and (4.13) may be denoted simply

$$\frac{A+G}{2} \;<\; E < \frac{4}{e}\frac{A+G}{2}, \tag{4.14}$$

$$\frac{2}{e}A \;<\; E < A. \tag{4.15}$$

The inequalities (4.14) and (4.15) make certain a bound of error that exponential mean E *are estimated by* A *or* $\dfrac{A+G}{2}$.

References

[1] C. GINI, Diuna formula comprensiva delle media, *Metron,* **13**(1938), 3–22.

[2] Guo Baini, Zhang Shiqin AND Qi Feng, An Elementary proof of Monotonicity for Extend Mean Values With Two Parameters, *Mathematics in Practice and Theory,* **29**(2)(1999), 169-173.

[3] Peter. A. HÄSTÖ, A Montonicity Property of Ratios of Symmetric Homogeneous Means. [ONLINE] Available online at `http://jipam-old.vu.edu.au/v3n5/013_02.pdf`.

[4] E. B. Leach AND M. C. Sholander, Extended mean values, *Amer. Math. Monthly,* **85**(1978), 84-90.

[5] E. B. Leach AND M. C. Sholander, Extended mean values, *J. Math. Anal. Appl.* , **92**(1983), 207-223.

[6] Qi Feng, Logarithmic convexities of the Extended Mean values, *J. London Math. Soc.* , to appear.

[7] Feng Qi, Generalized weighted mean values with two parameters, Proceedings of the Royal Society of London, Series,*Mathematical, Physical and Engineering Sciences,* **454**(1978) (1998): 2723-2732.

[8] F Qi, On a two-parameter family of nonhomogeneous mean values, *Tamkang Journal of Mathematics,* **29**(2)(1998), 155-163.

[9] Feng Qi, Generalized Abstracted Mean Values, *Journal of Inequalities in Pure and Applied Mathematics,* [ONLINE] Available online at http://jipam-old.vu.edu.au/v1n1/013_99.pdf.

[10] K. B. Stolarsky, Generalizations of the Logarithmic Mean, *Math. Mag.* , **48**(1975), 87-92.

[11] Sun Mingbao, Inequality for Two-Parameter Mean of Convex Function, *Mathematics in Practice and Theory,* **27**(3)(1997), 193-197.

[12] Shi Minqi AND Shi Huannan, An Upper Bound for Extend Logarithmic Mean, *Journal OF Mathematics,* **5**(1997), 37-38.

[13] Yang Zhenhang, Exponential Mean and Logarithmic Mean, *Mathematics in Practice and Theory,* **4**(1987), 76-78.

[14] Yang Zhenhang, Inequalities for Power Mean of Convex Function, *Mathematics in Practice and Theory,* **20**(1)(1990), 93-96

[15] Yang Zhenhang, Inequalities for General Mean of Convex Function, *Mathematics in Practice and Theory,* **33**(8)(2003), 136- 141.

[16] Yang Zhenhang, Simple Discriminance for Convexity of Homogeneous Functions and Applications, *Study in College Mathematics,* **7**(4)(2004), 14-19.

[17] Pales Z, Inequalities for differences of powers, *J. Math. Anal. Appl.* , **131**(1988), 271-281.

Zhejiang Electric Power Vocational Technical College, Hangzhou, Zhejiang, China, 311600

E-mail address: yzhkm@163.com