ON THE LOGARITHMICALLY CONVEXITY FOR TWO-PARAMETERS HOMOGENEOUS FUNCTIONS

ZHEN-HANG YANG

Abstract. Supposing $f(x,y)$ is a positive homogeneous function defined on $\mathbb{U}(\subseteq \mathbb{R}_+ \times \mathbb{R}_+)$, denoted the two-parameter homogeneous functions by $\mathcal{H}_f(a,b;p,q) = \left[\frac{f(a^p,b^p)}{f(x^q,b^q)}\right]^{\frac{1}{p-q}}$. If $f(x,y)$ is 3rd differentiablec, then the logarithmically convexity with respect to paraters p and q of $\mathcal{H}_f(p,q)$ depend on the sign of $J = (x-y)(xI_1)_x$, where $I_1 = (\ln f)_{xy}$. As applications of this results, a group of inequalities chains for homogeneous mean are established to generalize, strengthen and unify Ling Tong-po and Stolarsky inequalities; An conversed inequality chains for exponential (idential) mean is derived, which contains a reversed Stolarsky inequality; Several estimates of lower and upper bounds of two-parameter L-mean (extended mean) are presented.

1. Intruduction

In [14], the conception of two-parameter homogeneous function was introduced, its monotonicity was studied. For convenience, we quote it as follows

Definition 1. *Assume $f\colon \mathbb{U}(\subseteq \mathbb{R}_+ \times \mathbb{R}_+) \to \mathbb{R}_+$ is a homogeneous function for variable x and y, and is continuous and exist 1st partial derivative, $(a,b) \in \mathbb{R}_+ \times \mathbb{R}_+$ and $a \neq b$, $(p,q) \in \mathbb{R} \times \mathbb{R}$. If $(1,1) \notin \mathbb{U}$, then define that*

$$(1.1)\ \mathcal{H}_f(a,b;p,q) = \left[\frac{f(a^p,b^p)}{f(a^q,b^q)}\right]^{\frac{1}{p-q}} (p \neq q, pq \neq 0),$$

$$(1.2)\ \mathcal{H}_f(a,b;p,p) = \lim_{q\to p}\mathcal{H}_f(a,b;p,q) = G_f^{\frac{1}{p}}(a^p,b^p)(p = q \neq 0),$$

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where

$$G_f(x,y) = \exp\left[\frac{xf_x(x,y)\ln x + yf_y(x,y)\ln y}{f(x,y)}\right], \tag{1.3}$$

$f_x(x,y)$ or $f_y(x,y)$ denotes a partial derivative with respect to 1st or 2nd variable of $f(x,y)$ respectively.

If $(1,1) \in \mathbb{U}$, then define further

$$\mathcal{H}_f(a,b;p,0) = \left[\frac{f(a^p,b^p)}{f(1,1)}\right]^{\frac{1}{p}} (p \neq 0, q = 0), \tag{1.4}$$

$$\mathcal{H}_f(a,b;0,q) = \left[\frac{f(a^q,b^q)}{f(1,1)}\right]^{\frac{1}{q}} (p = 0, q \neq 0), \tag{1.5}$$

$$\mathcal{H}_f(a,b;0,0) = \lim_{p\to 0}\mathcal{H}_f(a,b;p,0) = G_{f,0}(a,b)(p = q = 0). \tag{1.6}$$

In the case of not being confused, we set

$$\mathcal{H}_f = \mathcal{H}_f(p,q) = \mathcal{H}_f(a,b;p,q),$$
$$G_{f,p} = G_{f,p}(a,b) = G_f^{\frac{1}{p}}(a^p,b^p) = \mathcal{H}_f(p,p)$$

It is no doubt that the conception of two-parameter homogeneous functions have developed greatly the extension of conception of two-parameter mean or extended mean and Gini mean.

As special cases of the two-parameter homogeneous function, the extended mean and Gini mean have been researched by various authors in [1–12, 15]. It is worth mentioning that Qi Feng studied the logarithmically convexity for the parameters of the extended mean in [5], and pointed out the two-parameters mean is a logarithmically concave function for two parameters on interval $(0,+\infty)$ and is a logarithmically convex function on interval $(-\infty,0)$. This is a very interesting and more useful result.

The aim of this paper is to investgate the logarithmically convexity with respect to the parameters of the two-parameter homogeneous function, and get the following results:

Theorem 1. *Let $f(x,y)$ be a positive n-order homogenous function defined on $\mathbb{U}(\subseteqq\mathbb{R}_+ \times \mathbb{R}_+)$, and be 3rd differentiable. If*

$$J = (x-y)(xI_1)_x \underset{(>)}{<} 0, \text{ where } I_1 = (\ln f)_{xy}, \tag{1.7}$$

then when $p,q \in (0,+\infty), \mathcal{H}_f(p,q)$ is logarithmically convex (concave) strictly for p or q respectively; while $p,q \in (-\infty,0), \mathcal{H}_f(p,q)$ is logarithmically concave (convex) strictly for p or q respectively.

Corollary 1. *The conditions is the same as Theorem 1.1's. If (1.7) holds then $\mathcal{H}_f(p,1-p)$ is strictly monotone decreasing (increasing) in $p \in (0,\frac{1}{2})$; $\mathcal{H}_f(p,1-p)$ is strictly monotone increasing (decreasing) in $p \in (\frac{1}{2},1)$.*

Corollary 2. *The conditions is the same as Theorem 1.1's. If (1.7) holds, then for $p, q \in (0, +\infty)$ with $p \neq q$, there is*

$$G_{f,\frac{p+q}{2}} \underset{(>0)}{<} \mathcal{H}_f(p,q) \underset{(>0)}{<} \sqrt{G_{f,p}G_{f,q}}. \tag{1.8}$$

For $p, q \in (-\infty, 0)$ with $p \neq q$, the inequality (1.8) reverses.

2. Proof of Main Results

For proving Theorem 1 and Corollary 1 and 2, we need to the following lemmas, Lemma 1 and 2 of which are from section 3 in [13].

Lemma 1. *Let $f(x,y), g(x,y)$ is a n, m-order homogenous function over Ω respectively, then $f \cdot g, f/g (g \neq 0)$ is a $n+m, n-m$-order homogenous function over Ω respectively.*

If for a certain p and $(x^p, y^p) \in \Omega$, $f^p(x,y)$ exist, then $f(x^p, y^p), f^p(x,y)$ are both np-order homogeneous functions over Ω.

Lemma 2. *Let $f(x,y)$ be a n-order homogeneous function over Ω, and f_x, f_y both exist, then f_x, f_y are both $n-1$-order homogeneous function over Ω, furthermore we have*

$$xf_x + yf_y = nf. \tag{2.1}$$

In particular, when $n = 1$ and $f(x,y)$ is 1st differentiable over Ω, then

$$xf_x + yf_y = f, \tag{2.2}$$

$$xf_{xx} + yf_{xy} = 0, \tag{2.3}$$

$$xf_{xy} + yf_{yy} = 0. \tag{2.4}$$

Lemma 3. *Let $f(x,y)$ be a positive n-order homogenous function defined on $\mathbb{U}(\subseteqq \mathbb{R}_+ \times \mathbb{R}_+)$, and be 2nd differentiable. Setting $T(t) = \ln f(a^t, b^t), x = a^t, y = b^t, a, b > 0$, then*

$$T'(t) = \ln G_f^{\frac{1}{t}}(a^t, b^t), \tag{2.5}$$

$$T''(t) = -xyI_1(\ln b - \ln a)^2, \text{ where } I_1 = (\ln f)_{xy}. \tag{2.6}$$

Proof. First, by (1.3) and direct calculating, we have

$$\begin{aligned} T'(t) &= \frac{a^t f_x(a^t, b^t) \ln a + b^t f_y(a^t, b^t) \ln b}{f(a^t, b^t)} \\ &= \frac{1}{t} \frac{a^t f_x(a^t, b^t) \ln a^t + b^t f_y(a^t, b^t) \ln b^t}{f(a^t, b^t)} = \ln G_f^{\frac{1}{t}}(a^t, b^t). \end{aligned}$$

Second, since $f(x,y)$ is a positive n-order homogeneous function, from expression (2.2), we can obtain $x(\ln f)_x + y(\ln f)_y = n$ or

$$x(\ln f)_x = n - y(\ln f)_y, y(\ln f)_y = n - x(\ln f)_x.$$

And then

$$\begin{aligned} T'(t) &= \frac{a^t f_x(a^t,b^t)\ln a + b^t f_y(a^t,b^t)\ln b}{f(a^t,b^t)} \\ &= \frac{x f_x(x,y)\ln a + y f_y(x,y)\ln b}{f(x,y)} \\ &= x(\ln f)_x \ln a + y(\ln f)_y \ln b. \end{aligned}$$

Hence

$$\begin{aligned} T''(t) &= \frac{\partial T'(t)}{\partial x}\frac{dx}{dt} + \frac{\partial T'(t)}{\partial y}\frac{dy}{dt} \\ &= [y(\ln f)_y(\ln b - \ln a) + n\ln a]_x\, a^t \ln a + \\ &\quad [x(\ln f)_x(\ln a - \ln b) + n\ln b]_y\, b^t \ln b \\ &= y(\ln f)_{yx}(\ln b - \ln a)x\ln a + x(\ln f)_{xy}(\ln a - \ln b)y\ln b \\ &= -xy(\ln f)_{xy}(\ln b - \ln a)^2 \\ &= -xyI_1(\ln b - \ln a)^2. \end{aligned}$$

∎

Lemma 4. *Let $f(x,y)$ be a positive n -order homogenous function defined on $\mathbb{U}(\subseteqq \mathbb{R}_+ \times \mathbb{R}_+)$, and be 3rd differentiable, then*

$$T'''(t) = -Ct^{-3}J,$$

where $J = (x-y)(xI_1)_x$,$C = xy(x-y)^{-1}(\ln x - \ln y)^3 > 0$.

Proof. From Lemma 1 and 2, we can understand that $I_1 = (\ln f)_{xy} = (ff_{xy} - f_x f_y)/f^2$ is a -2-order homogeneous function of x and y, thus xyI_1 is a 0-order homogeneous function. By (2.1), we get

$$x(xyI_1)_x + y(xyI_1)_y = 0, \text{ or } \quad y(xyI_1)_y = -x(xyI_1)_x. \tag{2.7}$$

By Lemma 3 and notice $x = a^t, y = b^t$, and then

$$\begin{aligned}
T'''(t) &= \frac{dT''(t)}{dt} = \frac{d\left(-xyI_1(\ln b - \ln a)^2\right)}{dt} \\
&= -(\ln b - \ln a)^2 \left(\frac{\partial(xyI_1)}{\partial x}\frac{dx}{dt} + \frac{\partial(xyI_1)}{\partial y}\frac{dy}{dt}\right) \\
&= -(\ln b - \ln a)^2 \left[a^t \ln a \cdot (xyI_1)_x + b^t \ln b \cdot (xyI_1)_y\right] \\
&= -(\ln b - \ln a)^2 \left[(x(xyI_1)_x \ln a + y \ln b (xyI_1)_y \ln b)\right] \\
&= -(\ln b - \ln a)^2 \left(x(xyI_1)_x\right)(\ln a - \ln b) \\
&= (\ln b - \ln a)^3 xy(xI_1)_x \\
&= xy\frac{(\ln b - \ln a)^3}{x-y}\left[(x-y)(xI_1)_x\right] \\
&= xy\frac{(\ln x - \ln y)^3}{t^3(x-y)}\left[(x-y)(xI_1)_x\right] \\
&= -Ct^{-3}J.
\end{aligned}$$

∎

Next then we will follow on proving Theorem 1 and Corollary 1-2.

Proof. It needs only to prove the convexity for p of $\ln \mathcal{H}_f$.

1) when $p \neq q$, $\ln \mathcal{H}_f = \dfrac{T(p) - T(q)}{p - q}$,

$$\frac{\partial \ln \mathcal{H}_f}{\partial p} = \frac{(p-q)T'(p) - T(p) + T(q)}{(p-q)^2} = \frac{g(p)}{(p-q)^2}, \tag{2.8}$$

$$\frac{\partial^2 \ln \mathcal{H}_f}{\partial p^2} = \frac{(p-q)g'(p) - 2g(p)}{(p-q)^3} = \frac{k(p)}{(p-q)^3}, \tag{2.9}$$

where $g(p) = (p-q)T'(p) - T(p) + T(q)$, $k(p) = (p-q)g'(p) - 2g(p)$.

Since $g'(p) = (p-q)T"(p)$ and $g(q) = 0$, so $k(p) = (p-q)^2T''(p) - 2g(p)$ and $k(q) = 0$, and then

$$k'(p) = 2(p-q)T''(p) + (p-q)^2T'''(p) - 2g'(p) = (p-q)^2T'''(p). \tag{2.10}$$

By Mean-value Theorem, exist $\xi = q + \theta(p-q)$ with $\theta \in (0,1)$, such that

$$\begin{aligned}
\frac{\partial^2 \ln \mathcal{H}_f}{\partial p^2} &= \frac{k(p) - k(q)}{(p-q)^3} = \frac{k'(\xi)}{(p-q)^2} \\
&= \frac{(\xi - q)^2 T'''(\xi)}{(p-q)^2} = (1-\theta)^2 T'''(\xi).
\end{aligned} \tag{2.11}$$

It is obvious that the logarithmically convexity of $\ln \mathcal{H}_f$ depends on the sigh of $T'''(\xi)$. From Lemma 4, $T'''(\xi) = -C\xi^{-3}J, C > 0$.

So if $J=(x-y)(xI_1)_x<0$, when $p,q\in(0,+\infty)$, $\xi=q+\theta(p-q)>0$ with $\theta\in(0,1)$, we have $T'''(\xi)>0$; while $p,q\in(-\infty,0)$, that $T'''(\xi)<0$. It is reversed when $J=(x-y)(xI_1)_x>0$.

2) when $p=q$, by (2.5)

$$\ln\mathcal{H}_f=\ln G_f^{\frac{1}{p}}(a^p,b^p)=\frac{xf_x(x,y)\ln x+yf_y(x,y)\ln y}{f(x,y)}=T'(p), \tag{2.12}$$

where $x=a^p,y=b^p$. So

$$\frac{\partial^2\ln\mathcal{H}_f}{\partial p^2}=T'''(p)=-Cp^{-3}J.$$

where $J=(x-y)(xI_1)_x, C>0$.

Hence if $J=(x-y)(xI_1)_x<0$, then when $p\in(0,+\infty)$, that $T'''(\xi)>0$; while $p\in(-\infty,0)$, $T'''(\xi)<0$. It is converse when $J=(x-y)(xI_1)_x>0$.

Combining 1) with and 2), we complete the proof of this Theorem immediately. ∎

proof of Corollary 1.1. It prove only the case when $J=(x-y)(xI_1)_x<0$.

1) When $p\in(\frac{1}{2},1)$.Assume $p,q\in(\frac{1}{2},1)$ and $q<p$, by Theorem 1, $\mathcal{H}_f(p,q)$ is logarithmically convex for $p,q\in(0,+\infty)$, so exist α , $\beta>0$ with $\alpha+\beta=1$, such that

$$\mathcal{H}_f^{\alpha}(1-q,1-p)\mathcal{H}_f^{\beta}(p,1-p)>\mathcal{H}_f(\alpha(1-q)+\beta p,1-p).$$

Taking $\alpha=\dfrac{p-q}{p+q-1}$, $\beta=\dfrac{2q-1}{p+q-1}$, then $\alpha(1-q)+\beta p=q$, and then from the above expression, we obtain

$$\begin{aligned}
&\mathcal{H}_f^{\beta}(p,1-p)>\mathcal{H}_f(q,1-p)\mathcal{H}_f^{-\alpha}(1-q,1-p)\\
&=\left[\frac{f(a^q,b^q)}{f(a^{1-p},b^{1-p})}\right]^{\frac{1}{p+q-1}}\left[\frac{f(a^{1-q},b^{1-q})}{f(a^{1-p},b^{1-p})}\right]^{\frac{1}{p-q}\cdot\frac{-(p-q)}{p+q-1}}\\
&=\left[\frac{f(a^q,b^q)}{f(a^{1-p},b^{1-p})}\right]^{\frac{1}{p+q-1}}\left[\frac{f(a^{1-q},b^{1-q})}{f(a^{1-p},b^{1-p})}\right]^{\frac{-1}{p+q-1}}\\
&=\left[\frac{f(a^q,b^q)}{f(a^{1-p},b^{1-p})}\frac{f(a^{1-p},b^{1-p})}{f(a^{1-q},b^{1-q})}\right]^{\frac{1}{p+q-1}}\\
&=\left[\frac{f(a^q,b^q)}{f(a^{1-q},b^{1-q})}\right]^{\frac{1}{p+q-1}}\\
&=\mathcal{H}_f^{\frac{2q-1}{p+q-1}}(q,1-q)\\
&=\mathcal{H}_f^{\beta}(q,1-q).
\end{aligned}$$

Extract the β power root of two sides, then $\mathcal{H}_f(p,1-p)>\mathcal{H}_f(q,1-q)$, namely when $p\in(\frac{1}{2},1)$, $\mathcal{H}_f(p,1-p)$ is strictly monotone increasing for p.

When $p \in (0, \frac{1}{2})$. Assume $p, q \in (0, \frac{1}{2})$ and $q < p$, notice $1-p, 1-q \in (\frac{1}{2}, 1)$ and $1-p < 1-q$, so there is $\mathcal{H}_f(1-p, p) < \mathcal{H}_f(1-q, q)$, i.e. $\mathcal{H}_f(p, 1-p) < \mathcal{H}_f(q, 1-q)$. It shows that $\mathcal{H}_f(p, 1-p)$ is strictly monotone decreasing for p. ∎

proof of Corollary 1.2. From Definition 1.1 and (2.12), we understand that

$$\begin{aligned} \ln \mathcal{H}_f(p,q) &= \frac{1}{p-q} \ln \frac{f(a^p, b^p)}{f(a^q, b^q)} = \frac{T(p) - T(q)}{p-q} = \frac{1}{p-q} \int_q^p T'(t) dt \\ &= \frac{1}{p-q} \int_q^p \ln G_f^{\frac{1}{t}}(a^t, b^t) dt = \frac{1}{p-q} \int_q^p \ln G_{f,t} dt. \end{aligned} \tag{2.13}$$

From Theorem 1, if $J = (x-y)(xI_1)_x < 0$, then $\ln G_{f,t}$ is strictly convex for $t \in (0, +\infty)$, and is concave strictly for $t \in (-\infty, 0)$. So when $p, q \in (0, +\infty)$, by using well-known Hermite-Hadamard inequality, we have

$$\ln G_{f, \frac{p+q}{2}} < \frac{1}{p-q} \int_q^p \ln G_{f,t} \mathrm{d}t < \frac{\ln G_{f,p} + \ln G_{f,q}}{2}, \tag{2.14}$$

i.e. inequality (1.8) holds. When $p, q \in (-\infty, 0)$, (2.14) is reverse, and inequality (1.8) is also reverse with it. Obviously, if $J = (x-y)(xI_1)_x > 0$, then the conclusions are reversed. ∎

3. Some Conclusions and Applications

By Theorem 1, the logarithmically convexity of $\mathcal{H}_f$ depends on the sign of $J = (x-y)(xI_1)_x$. Combining Theorem 1 with Corollary 1 and 2, we can get some conclusions about logarithmically convexity of $\mathcal{H}_f$, where $f(x,y) = L(x,y)$,$A(x,y)$,$E(x,y)$,$D(x,y)$. From it we will present a series of new inequalities concerning logarithm mean, exponential mean, power-exponential mean and exponential-geometry mean, meanwhile propose estimations of upper and lower bounds of two-parameter L-mean.

Case 1. $f(x,y) = L(x,y) = \dfrac{x-y}{\ln x - \ln y} (x, y > 0, x \neq y)$,

$$\mathcal{H}_L(a,b;p,q) = \begin{cases} \left(\dfrac{q(a^p - b^p)}{p(a^q - b^q)} \right)^{\frac{1}{p-q}} & p \neq q, pq \neq 0 \\ L^{\frac{1}{p}}(a^p, b^p) & p \neq 0, q = 0 \\ L^{\frac{1}{q}}(a^q, b^q) & p = 0, q \neq 0 \\ G_{L,p}(a,b) & p = q \neq 0 \\ G(a,b) & p = q = 0 \end{cases}, \tag{3.1}$$

where $G_{L,p}(a,b) = E_p(a,b) = E^{\frac{1}{p}}(a^p, b^p) = E_p, E(a,b) = e^{-1}\left(\frac{a^a}{b^b}\right)^{\frac{1}{a-b}}$, $G(a,b) = \sqrt{ab}$.

$$\begin{aligned} I_1 &= (\ln f)_{xy} = \frac{1}{(x-y)^2} - \frac{1}{xy(\ln x - \ln y)^2}, \\ J &= (x-y)(xI_1)_x = (x-y)\left[-\frac{x+y}{(x-y)^3} + \frac{2}{xy(\ln x - \ln y)^3}\right] \\ &= \frac{2}{xy(x-y)^2}\left[L^3(x,y) - \frac{x+y}{2}\left(\sqrt{xy}\right)^2\right]. \end{aligned}$$

It follows that well-known inequality $L(x,y) > \left(\frac{x+y}{2}\right)^{\frac{1}{3}}\left(\sqrt{xy}\right)^{\frac{2}{3}}$, $J > 0$.

Case 2. $f(x,y) = A(x,y) = \frac{x+y}{2} (x, y > 0)$,

$$\mathcal{H}_A(a,b;p,q) = \begin{cases} \left(\frac{a^p + b^p}{a^q + b^q}\right)^{\frac{1}{p-q}} & p \neq q \\ G_{A,p}(a,b) & p = q \neq 0 \\ G(a,b) & p = q = 0 \end{cases}, \tag{3.2}$$

where $G_{A,p}(a,b) = Z_p(a,b) = Z^{\frac{1}{p}}(a^p, b^p) = Z_p.Z(a,b) = a^{\frac{a}{a+b}} b^{\frac{b}{a+b}}$.

$$\begin{aligned} I_1 &= (\ln f)_{xy} = -\frac{1}{(x+y)^2}, \\ J &= (x-y)(xI_1)_x = \frac{(x-y)^2}{(x+y)^3} > 0. \end{aligned}$$

Case 3. $f(x,y) = E(x,y) = e^{-1}\left(\frac{x^x}{y^y}\right)^{\frac{1}{x-y}} (x, y > 0, x \neq y)$,

$$\mathcal{H}_E(a,b;p,q) = \begin{cases} \left(\frac{E(a^p, b^p)}{E(a^q, b^q)}\right)^{\frac{1}{p-q}} & p \neq q \\ G_{E,p}(a,b) & p = q \neq 0 \\ G(a,b) & p = q = 0 \end{cases}, \tag{3.3}$$

where $G_{E,p}(a,b) = Y_p(a,b) = Y^{\frac{1}{p}}(a^p, b^p) = Y_p.\ Y(a,b) = Ee^{1-\frac{G^2}{L^2}}$.

$$\begin{aligned} I_1 &= (\ln f)_{xy} = \frac{1}{(x-y)^3}\left[2(x-y) - (x+y)(\ln x - \ln y)\right], \\ J &= (x-y)(xI_1)_x = \frac{1}{(x-y)^2}\left[-3(x^2-y^2) + (x^2 + 4xy + y^2)(\ln x - \ln y)\right] \\ &= -\frac{6(\ln x - \ln y)}{(x-y)^3}\left[\frac{x^2-y^2}{\ln x^2 - \ln y^2} - \frac{\frac{x^2+y^2}{2} + 2xy}{3}\right]. \end{aligned}$$

It follows that well-known inequality $L(x,y) < \dfrac{\frac{x+y}{2} + 2\sqrt{xy}}{3}$, $J > 0$.

Case 4. $f(x,y) = D(x,y) = |x-y| (x,y > 0, x \neq y)$,

$$\mathcal{H}_D(a,b;p,q) = \begin{cases} \left|\dfrac{a^p - b^p}{a^q - b^q}\right|^{\frac{1}{p-q}} & p \neq q, pq \neq 0 \\ G_{D,p}(a,b) & p = q \neq 0 \end{cases}, \tag{3.4}$$

where $G_{D,p}(a,b) = G_{D,p} = e^{\frac{1}{p}} E^{\frac{1}{p}}(a^p, b^p) = e^{\frac{1}{p}} E_p$.

$$\begin{aligned} I_1 &= (\ln f)_{xy} = \frac{1}{(x-y)^2}, \\ J &= (x-y)(xI_1)_x = -\frac{x+y}{(x-y)^2} < 0. \end{aligned}$$

Applying mechanically Theorem 1, Corollary 1 and 2, we immediately obtain the following

Conclusion 1. *For* $f(x,y) = L(x,y)$, $A(x,y)$, $E(x,y)$,

1) when $p, q \in (0, +\infty), \mathcal{H}_f(p,q)$ *is logarithmically concave strictly for* p *or* q *respectively; while* $p, q \in (-\infty, 0), \mathcal{H}_f(p,q)$ *is logarithmically convex strictly for* p *or* q *respectively.*

2) $\mathcal{H}_f(p, 1-p)$ *is strictly monotone increasing for* $p \in (0, \frac{1}{2})$; $\mathcal{H}_f(p, 1-p)$ *is strictly monotone decreasing for* $p \in (\frac{1}{2}, 1)$.

3) If $p, q \in (0, +\infty)$, *there is*

$$G_{f, \frac{p+q}{2}} > \mathcal{H}_f(p,q) > \sqrt{G_{f,p} G_{f,q}}. \tag{3.5}$$

Inequality (3.5) is reverse if $p, q \in (-\infty, 0)$.

Conclusion 2. *1) when* $p, q \in (0, +\infty), \mathcal{H}_D(p,q)$ *is logarithmically convex strictly for* p *or* q *respectively; while* $p, q \in (-\infty, 0), \mathcal{H}_D(p,q)$ *is logarithmically concave strictly for* p *or* q *respectively.*

2) $\mathcal{H}_D(p, 1-p)$ *is strictly monotone decreasing for* $p \in (0, \frac{1}{2})$; $\mathcal{H}_D(p, 1-p)$ *is strictly monotone increasing for* $p \in (\frac{1}{2}, 1)$.

3) If $p, q \in (0, +\infty)$, *there is*

$$G_{D, \frac{p+q}{2}} < \mathcal{H}_D(p,q) < \sqrt{G_{D,p} G_{D,q}}. \tag{3.6}$$

Inequality (3.6) reverses if $p, q \in (-\infty, 0)$.

By applying above conclusions, we will get some new inequalities.

Example 1. *A group of inequality chains for homogeneous mean*. *By 2) of Conclusion 1,* $\mathcal{H}_f(p, 1-p)$ *is strictly monotone decreasing for* $p \in (\frac{1}{2}, 1)$. *So there is*

$$\mathcal{H}_f(1,0) < \mathcal{H}_f(\frac{4}{5}, \frac{1}{5}) < \mathcal{H}_f(\frac{3}{4}, \frac{1}{4}) < \mathcal{H}_f(\frac{2}{3}, \frac{1}{3}) < \mathcal{H}_f(\frac{3}{5}, \frac{2}{5}) < \mathcal{H}_f(\frac{1}{2}, \frac{1}{2}),$$

i.e.

$$\begin{aligned}\frac{f(a,b)}{f(1,1)} &< \left[\frac{f(a^{\frac{4}{5}},b^{\frac{4}{5}})}{f(a^{\frac{1}{5}},b^{\frac{1}{5}})}\right]^{\frac{5}{3}} < \left[\frac{f(a^{\frac{3}{4}},b^{\frac{3}{4}})}{f(a^{\frac{1}{4}},b^{\frac{1}{4}})}\right]^{2} \\ &< \left[\frac{f(a^{\frac{2}{3}},b^{\frac{2}{3}})}{f(a^{\frac{1}{3}},b^{\frac{1}{3}})}\right]^{3} < \left[\frac{f(a^{\frac{3}{5}},b^{\frac{3}{5}})}{f(a^{\frac{2}{5}},b^{\frac{2}{5}})}\right]^{5} < a^{\frac{\sqrt{a}f_x(\sqrt{a},\sqrt{b})}{f(\sqrt{a},\sqrt{b})}} b^{\frac{\sqrt{b}f_y(\sqrt{a},\sqrt{b})}{f(\sqrt{a},\sqrt{b})}}. \end{aligned} \tag{3.7}$$

1) For $f(x,y)=L(x,y)$, notice $f(1,1)=1$, we get

$$\begin{aligned} L(a,b) &< \left(\frac{\frac{1}{5}(b^{\frac{4}{5}}-a^{\frac{4}{5}})}{\frac{4}{5}(b^{\frac{1}{5}}-a^{\frac{1}{5}})}\right)^{\frac{5}{3}} < \left(\frac{\frac{1}{4}(b^{\frac{3}{4}}-a^{\frac{3}{4}})}{\frac{3}{4}(b^{\frac{1}{4}}-a^{\frac{1}{4}})}\right)^{2} \\ &< \left(\frac{\frac{1}{3}(b^{\frac{2}{3}}-a^{\frac{2}{3}})}{\frac{2}{3}(b^{\frac{1}{3}}-a^{\frac{1}{3}})}\right)^{3} < \left(\frac{\frac{2}{5}(b^{\frac{3}{5}}-a^{\frac{3}{5}})}{\frac{3}{5}(b^{\frac{2}{5}}-a^{\frac{2}{5}})}\right)^{5} < E^2(\sqrt{a},\sqrt{b}), \end{aligned}$$

i.e

$$\begin{aligned} L(a,b) &< \left(\frac{(b^{\frac{1}{5}}+a^{\frac{1}{5}})(b^{\frac{2}{5}}+a^{\frac{2}{5}})}{4}\right)^{\frac{5}{3}} < \left(\frac{b^{\frac{1}{2}}+a^{\frac{1}{4}}b^{\frac{1}{4}}+a^{\frac{1}{2}}}{3}\right)^{2} \\ &< \left(\frac{b^{\frac{1}{3}}+a^{\frac{1}{3}}}{2}\right)^{3} < \left(\frac{2(b^{\frac{2}{5}}+b^{\frac{1}{5}}a^{\frac{1}{5}}+a^{\frac{2}{5}})}{3(b^{\frac{1}{5}}+a^{\frac{1}{5}})}\right)^{5} < E^2(\sqrt{a},\sqrt{b}). \end{aligned}$$

The above inequalities chain may be simply denoted by

$$L < M_{\frac{1}{5}}^{\frac{1}{3}} M_{\frac{2}{5}}^{\frac{2}{3}} < h_{\frac{1}{2}} < M_{\frac{1}{3}} < h_{\frac{2}{5}}^{2} M_{\frac{1}{5}}^{-1} < E_{\frac{1}{2}}, \tag{3.8}$$

where $M_p = (\frac{a^p+b^p}{2})^{\frac{1}{p}}$, $E_p = E^{\frac{1}{p}}(a^p,b^p)$, $h_p = \left[\frac{a^p+(\sqrt{ab})^p+b^p}{3}\right]^{\frac{1}{p}}$.

That $L < M_{\frac{1}{3}}$ is well-known Ling Tong-Po inequality. The above inequalities chain shows that can be inserted $M_{\frac{1}{5}}^{\frac{1}{3}} M_{\frac{2}{5}}^{\frac{2}{3}}$ and $h_{\frac{1}{2}}$ between L and $M_{\frac{1}{3}}$, so (3.8) strengthens Ling Tong-Po inequality.

2) For $f(x,y)=A(x,y)$, notice $f(1,1)=1$, then

$$A < M_{\frac{4}{5}}^{\frac{4}{3}} M_{\frac{1}{5}}^{-\frac{1}{3}} < M_{\frac{3}{4}}^{\frac{3}{2}} M_{\frac{1}{4}}^{-\frac{1}{2}} < M_{\frac{2}{3}}^{2} M_{\frac{1}{3}}^{-1} < M_{\frac{3}{5}}^{3} M_{\frac{2}{5}}^{-2} < Z_{\frac{1}{2}}, \tag{3.9}$$

where $M_p = (\frac{a^p+b^p}{2})^{\frac{1}{p}}$, $Z_p = Z^{\frac{1}{p}}(a^p,b^p)$.

3) For $f(x,y)=E(x,y)$, notice $f(1,1)=1$, $\frac{E(a^2,b^2)}{E(a,b)} = Z(a,b)$, then

$$E < Z_{\frac{1}{5}}^{\frac{1}{3}} Z_{\frac{2}{5}}^{\frac{2}{3}} < E_{\frac{3}{4}}^{\frac{3}{2}} E_{\frac{1}{4}}^{-\frac{1}{2}} < Z_{\frac{1}{3}} < E_{\frac{3}{5}}^{3} E_{\frac{2}{5}}^{-2} < Y_{\frac{1}{2}}, \tag{3.10}$$

where $Z_p = Z^{\frac{1}{p}}(a^p,b^p)$, $E_p = E^{\frac{1}{p}}(a^p,b^p)$, $Y_p = Y^{\frac{1}{p}}(a^p,b^p)$.

Remark 1. *If replace a, b with a^2, b^2 in (3.8)-(3.10), then them may be rewritten into*

$$E > h^2_{\frac{4}{5}} M^{-1}_{\frac{2}{5}} > M_{\frac{2}{3}} > h > M^{\frac{1}{3}}_{\frac{2}{5}} M^{\frac{2}{3}}_{\frac{4}{5}} > L_2 = \sqrt{LA}, \tag{3.11}$$

$$Z > M^3_{\frac{6}{5}} M^{-2}_{\frac{4}{5}} > M^2_{\frac{4}{3}} M^{-1}_{\frac{2}{3}} > M^{\frac{3}{2}}_{\frac{3}{2}} M^{-\frac{1}{2}}_{\frac{1}{2}} > M^{\frac{4}{3}}_{\frac{8}{5}} M^{-\frac{1}{3}}_{\frac{2}{5}} > M_2, \tag{3.12}$$

$$Y > E^3_{\frac{6}{5}} E^{-2}_{\frac{4}{5}} > Z_{\frac{2}{3}} > E^{\frac{3}{2}}_{\frac{3}{2}} E^{-\frac{1}{2}}_{\frac{1}{2}} > Z^{\frac{1}{3}}_{\frac{2}{5}} Z^{\frac{2}{3}}_{\frac{4}{5}} > E_2 = \sqrt{EZ}. \tag{3.13}$$

That $E > M_{\frac{2}{3}}$ is well-known Stolarsky inequality. (3.11) indicates that can be inserted $h^2_{\frac{4}{5}} M^{-1}_{\frac{2}{5}}$ between E and $M_{\frac{2}{3}}$, so (3.11) strengthens Stolarsky inequality. It follows that Lin Tong-Po and Stolarsky inequality are unified into a the same inequality's chain and refined by (3.8) or (3.11). Meanwhile they are generalized the case of arithmetic mean and exponential mean by (3.9) or (3.12) and (3.10) or (3.13) in parallel. So we call (3.7) the homogeneous mean's L-S inequality chain.

Remark 2. *There include some simple and brand-new inequalities in (3.8)-(3.13), such as $Z > M_2$ from (3.12), i.e.*

$$Z > \sqrt{\frac{a^2 + b^2}{2}}. \tag{3.14}$$

While $Z > M^{\frac{3}{2}}_{\frac{3}{2}} M^{-\frac{1}{2}}_{\frac{1}{2}}$ may be transformed into $Z > \dfrac{a^{\frac{3}{2}} + b^{\frac{3}{2}}}{a^{\frac{1}{2}} + b^{\frac{1}{2}}} = a + b - \sqrt{ab}$, i.e. $\frac{Z+G}{2} > A$.

By (3.10) we can get

$$E < Z_{\frac{1}{3}} < Y_{\frac{1}{2}}, \tag{3.15}$$

or

$$Y > Z_{\frac{2}{3}} > E_2 = \sqrt{EZ}. \tag{3.16}$$

Example 2. ***An conversed inequality for exponential mean.*** *By 2) of Conclusion 2, noticed $D(1,1)$ does'nt exist, we have*

$$\mathcal{H}_D(\frac{1}{2}, \frac{1}{2}) < \mathcal{H}_D(\frac{3}{5}, \frac{2}{5}) < \mathcal{H}_D(\frac{2}{3}, \frac{1}{3}) < \mathcal{H}_D(\frac{3}{4}, \frac{1}{4}) < \mathcal{H}_D(\frac{4}{5}, \frac{1}{5}),$$

i.e.

$$\begin{aligned} e^2 E^2(\sqrt{a}, \sqrt{b}) &< \left(\frac{b^{\frac{3}{5}} - a^{\frac{3}{5}}}{b^{\frac{2}{5}} - a^{\frac{2}{5}}}\right)^5 < \left(\frac{b^{\frac{2}{3}} - a^{\frac{2}{3}}}{b^{\frac{1}{3}} - a^{\frac{1}{3}}}\right)^3 \\ &< \left(\frac{b^{\frac{3}{4}} - a^{\frac{3}{4}}}{b^{\frac{1}{4}} - a^{\frac{1}{4}}}\right)^2 < \left(\frac{b^{\frac{4}{5}} - a^{\frac{4}{5}}}{b^{\frac{1}{5}} - a^{\frac{1}{5}}}\right)^{\frac{5}{3}}, \end{aligned}$$

i.e.

$$e^2E^2(\sqrt{a},\sqrt{b}) < \left(\frac{b^{\frac{2}{5}}+b^{\frac{1}{5}}a^{\frac{1}{5}}+a^{\frac{2}{5}}}{b^{\frac{1}{5}}+a^{\frac{1}{5}}}\right)^5 < (b^{\frac{1}{3}}+a^{\frac{1}{3}})^3$$

$$< \left(b^{\frac{1}{2}}+a^{\frac{1}{4}}b^{\frac{1}{4}}+a^{\frac{1}{2}})\right)^2 < \left((b^{\frac{1}{5}}+a^{\frac{1}{5}})(b^{\frac{2}{5}}+a^{\frac{2}{5}})\right)^{\frac{5}{3}}. \tag{3.17}$$

If replace a, b with a^2, b^2 in (3.17), then which may be denoted simply by

$$E < \frac{\sqrt{486}}{8e}h^2_{\frac{4}{5}}M^{-1}_{\frac{2}{5}} < \frac{\sqrt{8}}{e}M_{\frac{2}{3}} < \frac{3}{e}h < \frac{\sqrt[3]{32}}{e}M^{\frac{1}{3}}_{\frac{2}{5}}M^{\frac{2}{3}}_{\frac{4}{5}} \tag{3.18}$$

(3.18) is a reversed inequality chain of five items in left side of (3.11).

Remark 3. *By (3.11) and (3.18), we get*

$$M^{\frac{1}{3}}_{\frac{2}{5}}M^{\frac{2}{3}}_{\frac{4}{5}} < h < M_{\frac{2}{3}} < h^2_{\frac{4}{5}}M^{-1}_{\frac{2}{5}} < E \tag{3.19}$$

$$< \frac{\sqrt{486}}{8e}h^2_{\frac{4}{5}}M^{-1}_{\frac{2}{5}} < \frac{\sqrt{8}}{e}M_{\frac{2}{3}} < \frac{3}{e}h < \frac{\sqrt[3]{32}}{e}M^{\frac{1}{3}}_{\frac{2}{5}}M^{\frac{2}{3}}_{\frac{4}{5}}.$$

From it, we have further

$$1 < E/M^{\frac{1}{3}}_{\frac{2}{5}}M^{\frac{2}{3}}_{\frac{4}{5}} < \sqrt[3]{32}/e \approx 1.16794, \tag{3.20}$$

$$1 < E/h < 3/e \approx 1.10364, \tag{3.21}$$

$$1 < E/M_{\frac{2}{3}} < \sqrt{8}/e \approx 1.04052, \tag{3.22}$$

$$1 < E/h^2_{\frac{4}{5}}M^{-1}_{\frac{2}{5}} < \sqrt{486}/8e \approx 1.01376. \tag{3.23}$$

Inequalities (3.20)-(3.23) indicate that regardless the size of positive numbers a and b, the relative error estimating exponential mean E by $M^{\frac{1}{3}}_{\frac{2}{5}}M^{\frac{2}{3}}_{\frac{4}{5}}$, h, $M_{\frac{2}{3}}$ and $h^2_{\frac{4}{5}}M^{-1}_{\frac{2}{5}}$ are approximate to 17%, 10%, 4%, and 1% respectively.

Example 3. ***Estimations of lower and upper bounds of two-parameter L-mean (or extended mean).*** *From 3) of Conclusion 2, and notice*

$$G_{D,p} = e^{\frac{1}{p}}E^{\frac{1}{p}}(x^p, y^p) = e^{\frac{1}{p}}E_p.$$

So we have

$$e^{\frac{2}{p+q}}E_{\frac{p+q}{2}} < \mathcal{H}_D(p,q) < \sqrt{e^{\frac{1}{p}}E_p e^{\frac{1}{q}}E_q}, \text{ if } p,q>0 \text{ and } p \neq q. \tag{3.24}$$

If notice further

(3.25)

$$\mathcal{H}_D(p,q) = \left|\frac{b^p-a^p}{b^q-a^q}\right|^{\frac{1}{p-q}} = \left(\frac{p}{q}\right)^{\frac{1}{p-q}}\left(\frac{q(b^p-a^p)}{p(b^q-a^q)}\right)^{\frac{1}{p-q}} = e^{\frac{1}{L(p,q)}}\mathcal{H}_L(p,q),$$

then (3.24) can be rewritten into

$$e^{\frac{1}{A(p,q)}-\frac{1}{L(p,q)}}E_{\frac{p+q}{2}} < \mathcal{H}_L(p,q) < e^{\frac{1}{H(p,q)}-\frac{1}{L(p,q)}}\sqrt{E_pE_q}, \tag{3.26}$$

where $A(p,q) = \dfrac{p+q}{2}$, $H(p,q) = \dfrac{2pq}{p+q}$, $L(p,q) = \dfrac{p-q}{\ln(p/q)}$,$p,q > 0$ *and* $p \neq q$.

Combining (3.5) with (3.26), we can get other two expressions of estimations of the two-parameter L-mean $\mathcal{H}_L(p,q)$.

$$e^{\frac{1}{A(p,q)}-\frac{1}{L(p,q)}}E_{\frac{p+q}{2}} < \mathcal{H}_L(p,q) < E_{\frac{p+q}{2}} \tag{3.27}$$

$$\sqrt{E_pE_q} < \mathcal{H}_L(p,q) < e^{\frac{1}{H(p,q)}-\frac{1}{L(p,q)}}\sqrt{E_pE_q} \tag{3.28}$$

where $p,q > 0$ *and* $p \neq q$. *the inequalities 3.8, 3.9 reverse if* $p,q < 0$ *and* $p \neq q$.

Lastly, we can find out some new inequalities by using the theorem and corollaries in this paper. Discuss no longer here.

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Zhejiang Electric Power Vocational Technical College, Hangzhou, Zhejiang, China, 311600
E-mail address: yzhkm@163.com