ON THE MONOTONICITY AND LOG-CONVEXITY FOR ONE-PARAMETER HOMOGENEOUS FUNCTIONS

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ABSTRACT. That $\mathcal{H}_{1f}(p) := \mathcal{H}_f(p, 1+p)$ is called one-parameter homogeneous functions. The monotonicity of $\mathcal{H}_{1f}(p)$ depends on the sign of $I_1 = (\ln f)_{xy}$; While the log-convexity of $\mathcal{H}_{1f}(p)$, the monotonicity of $\mathcal{H}_f(p, 1-p)$ and $\bar{\mathcal{H}}_{1f}(p) = \mathcal{H}_{1f}(p)\mathcal{H}_{1f}(-p)$ depend on the sign of $J = (x-y)(xI_1)_x$. By straightforward computations, some conclusions on the monotonicity of $\mathcal{H}_{1f}(p)$, $\mathcal{H}_f(p, 1-p)$, $\bar{\mathcal{H}}_{1f}(p)$ and log-convexity of $\mathcal{H}_{1f}(p)$ are presented, where $f(x,y) = L(x,y)$, $A(x,y)$, $E(x,y)$ and $D(x,y)$. As one of the special cases, Wing-Sum Cheung and Feng Qi's results are derived.

1. INTRODUCTION

The one-parameter mean values $J(p; a, b)$ (for avoiding confussion in notations, we replace $J(p; a, b)$ with $\mathcal{S}(p; a, b)$ in what follows) for $a \neq b$ are defined in [2, 13] and introduced in [7] by

$$\mathcal{S}(p; a, b) = \begin{cases} \frac{p(a^{p+1}-b^{p+1})}{(p+1)(a^p-b^p)}, & p \neq 0, -1; \\ \frac{a-b}{\ln a - \ln b}, & p = 0; \\ \frac{ab(\ln a - \ln b)}{a-b}, & p = -1. \end{cases} \tag{1.1}$$

and $\mathcal{S}(p; a, b)$ is strictly increasing in $p \in \mathbb{R}$.

In [6], the following results in [2, 3] by Alzer are mentioned:

1) When $p \neq 0$, we have

(1.2)
$$G(a,b) < \sqrt{\mathcal{S}(p; a, b)\mathcal{S}(-p; a, b)} < L(a,b) < \frac{\mathcal{S}(p; a, b) + \mathcal{S}(-p; a, b)}{2} < A(a,b);$$

2) For $a_1, a_2 >$ and $b_1, b_2 > 0$, if $p > 1$, then

$$\mathcal{S}(p; a_1 + a_2, b_1 + b_2) \leq \mathcal{S}(p; a_1, b_1) + \mathcal{S}(p; a_2, b_2); \tag{1.3}$$

if $p \leq 1$, inequality (1.3) is reversed.

3) If (a_1, b_1) and (a_2, b_2) are similarly or oppositely ordered, then, if $p < -\frac{1}{2}$, we have

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$$\mathcal{S}(r; a_1a_2, b_1b_2) \geq (\leq)\mathcal{S}(p; a_1, b_1)\mathcal{S}(p; a_2, b_2); \tag{1.4}$$

if $p \geq -\frac{1}{2}$, then inequality (1.4) is reversed.

4) For $a, b > 0$, if $p < q < r \leq -\frac{1}{2}$, then

$$[\mathcal{S}(q; a, b)]^{r-p}[\mathcal{S}(p; a, b)]^{r-q}[\mathcal{S}(r; a, b)]^{q-p}; \tag{1.5}$$

if $-\frac{1}{2} \leq p < q < r$, inequality (1.5) is reversed.

Moreover, H. Alzer in [3] raised a question about the convexity of $p \ln \mathcal{S}(p; a, b)$ and proved that $(p + 1)\mathcal{S}(p; a, b)$ is convex.

Wing-Sum Cheung and Feng Qi researched the log-convexity of the one-parameter mean values $\mathcal{S}(p; a, b)$ and the monotonicity of $\mathcal{S}(p)\mathcal{S}(-p)$ for $p \in \mathbb{R}$,and presented the following results (see [4]):

Theorem 1. *For fixed positive numbers a and b with $a \neq b$, then the one-parameter mean values $\mathcal{S}(p)$ defined by (1.1) are strictly log-convex in $(-\infty, -\frac{1}{2})$ and strictly log-concave in $(-\frac{1}{2}, +\infty)$.*

Theorem 2. *Let $\bar{\mathcal{S}}(p) = \mathcal{S}(p)\mathcal{S}(-p)$ with $p \in \mathbb{R}$ for fixed positive numbers a and b with $a \neq b$. Then the function $\bar{\mathcal{S}}(p)$ is strictly increasing in $(-\infty, 0)$ and strictly decreasing in $(0, +\infty)$.*

On the other hand, Zhen-Hang Yang also derived Minkowski, Hölder and Tchebchef type inequalities of $\mathcal{S}(p; a, b)$, by using simplified discriminance involving convexity of homogeneous functions in two variables deduced from the properties of homogeneous functions (see [14]).

Meanwhile the two-parameter homogeneous functions were introduced in [15]. That is:

Definition 1. *Assume $f\colon \mathbb{U}(\subseteqq \mathbb{R}_+ \times \mathbb{R}_+) \to \mathbb{R}_+$ is a homogeneous function for variable x and y, and is continuous and 1-time partial derivative exist, $(a, b) \in \mathbb{R}_+ \times \mathbb{R}_+$ with $a \neq b$, $(p, q) \in \mathbb{R} \times \mathbb{R}$. If $(1, 1) \notin U$, then define that*

$$\mathcal{H}_f(a, b; p, q) = \left[\frac{f(a^p, b^p)}{f(a^q, b^q)}\right]^{\frac{1}{p-q}} (p \neq q, pq \neq 0), \tag{1.6}$$

$$\mathcal{H}_f(a, b; p, p) = \lim_{q \to p} \mathcal{H}_f(a, b; p, q) = G_{f,p}(a, b)(p = q \neq 0). \tag{1.7}$$

where $G_{f,p}(a, b) = G_f^{\frac{1}{p}}(a^p, b^p)$,

$$G_f(x, y) = \exp\left[\frac{xf_x(x, y)\ln x + yf_y(x, y)\ln y}{f(x, y)}\right], \tag{1.8}$$

in which $f_x(x, y)$ and $f_y(x, y)$ denote 1st order partial derivative to 1st and 2nd variable of $f(x, y)$, respectively.

If $(1,1) \in U$*, then define further*

$$\mathcal{H}_f(a,b;p,0) = \left[\frac{f(a^p,b^p)}{f(1,1)}\right]^{\frac{1}{p}} (p \neq 0, q = 0), \tag{1.9}$$

$$\mathcal{H}_f(a,b;0,q) = \left[\frac{f(a^q,b^q)}{f(1,1)}\right]^{\frac{1}{q}} (p = 0, q \neq 0), \tag{1.10}$$

$$\mathcal{H}_f(a,b;0,0) = \lim_{p\to 0}\mathcal{H}_f(a,b;p,0) = a^{\frac{f_x(1,1)}{f(1,1)}} b^{\frac{f_y(1,1)}{f(1,1)}} (p = q = 0). \tag{1.11}$$

In the case of not being confused, we set

$$\begin{aligned} \mathcal{H}_f &= \mathcal{H}_f(p,q) = \mathcal{H}_f(a,b;p,q) = \left[\frac{f(p)}{f(q)}\right]^{\frac{1}{p-q}}, \\ G_{f,p} &= G_{f,p}(a,b) = G_f^{\frac{1}{p}}(a^p,b^p) = \mathcal{H}_f(p,p). \end{aligned}$$

The following properties of $\mathcal{H}_f(p,q)$ are obvious by some easy calculations:
Property 1 $\mathcal{H}_f(a,b;p,q)$ *are symmetric with respect to* a, b *and* p, q*, i.e.*

$$\mathcal{H}_f(a,b;p,q) = \mathcal{H}_f(a,b;q,p). \tag{1.12}$$

$$\mathcal{H}_f(a,b;p,q) = \mathcal{H}_f(b,a;p,q) \tag{1.13}$$

Property 2 Let

$$T(t) = \ln f(a^t,b^t) \tag{1.14}$$

then

$$T'(t) = \frac{a^t f_x(a^t,b^t)\ln a + b^t f_y(a^t,b^t)\ln b}{f(a^t,b^t)} = \ln G_f^{\frac{1}{t}}(a^t,b^t), \tag{1.15}$$

where $t \neq 0$ if $(1,1) \notin \mathbb{U}$.
Property 3 If $f(x,y) = f(y,x)$ for all $(x,y) \in \mathbb{U}$, then

$$\mathcal{H}_f(t,-t) = G^n, \tag{1.16}$$

$$T(t) - T(-t) = 2nt\ln G, \tag{1.17}$$

where $G = \sqrt{ab}$.

There are the following two results concerning the two-parameter homogeneous functions.

Theorem 3. *Let* $f(x,y)$ *be a positive* n*-order homogenous function defined on* $\mathbb{U}(\subseteqq \mathbb{R}_+ \times \mathbb{R}_+)$*, and be 2-time differentiable. If* $I_1 = (\ln f)_{xy} < (>)0$*, then* $\mathcal{H}_f(p,q)$ *is strictly increasing (decreasing) in* p *or* q*.*

Theorem 4. *Let* $f(x,y)$ *be a positive* n*-order homogenous function defined on* $\mathbb{U}(\subseteqq \mathbb{R}_+ \times \mathbb{R}_+)$*, and be 3-time differentiable. If*

$$J = (x-y)(xI_1)_x < (>)0, \text{ where } I_1 = (\ln f)_{xy}, \tag{1.18}$$

then $\mathcal{H}_f(p,q)$ *is strictly log-convex (log-concave) in* $p \in (0,+\infty)$*, while log-concave (log-convex) in* $p \in (-\infty,0)$*.*

For another parameter q*, the above conclusion is also true.*

Obviously, the one-parameter mean is only a special case of two-parameter mean. In the same way, let $q = 1+p$ in Definition 1, then the two-parameter homogeneous functions become the so-called one-parameter homogeneous functions.

The aim of this paper is to extend the one-parameter mean into the one-parameter homogeneous functions based on [15], and investigate its monotonicity and log-convexity in parameters further. As a special case, Theorem 1 and 2 will be deduced.

2. Basic Conception and Main Results

First we present the definition of the one-parameter homogeneous functions now.

Definition 2. *Let $q = 1 + p$ in the two-parameter homogeneous functions $\mathcal{H}_f(p, q)$, then call it one-parameter homogeneous functions, and denote by $\mathcal{H}_{1f}(p) = \mathcal{H}_f(p, 1 + p)$.*

From Definition 2, for $f(x, y) = L(x, y)$, $A(x, y)$, $E(x, y)$, and $D(x, y) = |x - y|$, we have

$$\mathcal{H}_{1L}(a,b;p) = \begin{cases} \frac{p(a^{p+1}-b^{p+1})}{(p+1)(a^p-b^p)}, & p \neq 0, -1; \\ L(a,b), & p = 0; \\ \frac{G^2(a,b)}{L(a,b)}, & p = -1. \end{cases} \tag{2.1}$$

$$\mathcal{H}_{1A}(a,b;p) = \frac{a^{p+1}+b^{p+1}}{a^p+b^p}. \tag{2.2}$$

$$\mathcal{H}_{1E}(a,b;p) = \frac{E(a^{p+1},b^{p+1})}{E(a^p,b^p)}. \tag{2.3}$$

$$\mathcal{H}_{1D}(a,b;p) = \left|\frac{x^{p+1}-y^{p+1}}{x^p-y^p}\right|, \quad p \neq 0. \tag{2.4}$$

That $\mathcal{H}_{1L}(a, b; p)$ is just the one-parameter mean of positive numbers a and b. To avoid to be confused, it is called one-parameter logarithmic mean; In the same way, we call $\mathcal{H}_{1A}(a, b; p)$ and $\mathcal{H}_{1E}(a, b; p)$ one-parameter arithmetic mean (also call Lehmer mean) and one-parameter exponential mean, respectively.

Since $D(x, y)$ is no a certain mean of positive numbers x and y, but a absolute value of difference function, so we call one-parameter homogeneous differnce function temporarily.

Concerning the monotonicity and log-convexity of the one-parameter homogeneous functions, there are the following main results.

Theorem 5. *Let $f(x, y)$ be a positive n-order homogenous function defined on $\mathbb{U}(\subseteqq \mathbb{R}_+ \times \mathbb{R}_+)$, and be 2-time differentiable. If $I_1 = (\ln f)_{xy} < (>)0$, then $\mathcal{H}_{1f}(p)$ is strictly increasing (decreasing) in $p \in (-\infty, 0) \cup (0, +\infty)$.*

Theorem 6. *Let $f(x, y)$ be a positive n-order homogenous function defined on $\mathbb{U}(\subseteqq \mathbb{R}_+ \times \mathbb{R}_+)$, and be 3-time differentiable. If $J = (x-y)(xI_1)_x < (>)0$, then*

1) $\mathcal{H}_{1f}(p)$ is strictly log-concave (log-convex) in $p \in (-\infty, -1)$, strictly log-convex (log-concave) in $p \in (0, +\infty)$.

2) If $f(x,y)$ satisfies $f(x,y) = f(y,x)$ further, then $\mathcal{H}_{1f}(p)$ is strictly log-concave (log-convex) in $p \in (-\infty, -\frac{1}{2})$, log-convex (log-concave) in $p \in (-\frac{1}{2}, 0) \cup (0, +\infty)$.

According to 2) of the Theorem 6, and the properties of convex functions, the functions $\frac{\ln \mathcal{H}_{1f}(p-1) - \ln \mathcal{H}_{1f}(-\frac{1}{2})}{p-1-(-\frac{1}{2})}$ is strictly decreasing (increasing) for $p - 1 \in (-\infty, -\frac{1}{2})$ and increasing (decreasing) for $p - 1 \in (-\frac{1}{2}, 0) \cup (0, +\infty)$ if $J = (x-y)(xI_1)_x < (>)0$.

Notice

$$\begin{aligned}\frac{\ln \mathcal{H}_{1f}(p-1) - \ln \mathcal{H}_{1f}(-\frac{1}{2})}{p-1-(-\frac{1}{2})} &= \frac{\ln f(p) - \ln f(p-1) - \ln \mathcal{H}_{1f}(-\frac{1}{2})}{p-1-(-\frac{1}{2})} \\ &= \frac{\ln f(p) - \ln f(1-p)}{p - \frac{1}{2}} = 2 \ln \mathcal{H}_f(p, 1-p),\end{aligned}$$

so we have the following:

Corollary 1. *Let $f(x,y)$ be a positive n-order homogenous function defined on $\mathbb{U}(\subseteqq \mathbb{R}_+ \times \mathbb{R}_+)$, and be 3-time differentiable, and satisfies $f(x,y) = f(y,x)$, further. If $J = (x-y)(xI_1)_x < (>)0$, then the function $\mathcal{H}_f(p, 1-p)$ is strictly decreasing (increasing) in $(-\infty, 0) \cup (0, \frac{1}{2})$, strictly increasing (decreasing) in $(\frac{1}{2}, +\infty)$, where*

$$\mathcal{H}_f(p, 1-p) = \begin{cases} \left(\frac{f(p)}{f(1-p)}\right)^{\frac{1}{2p-1}}, & p \neq \frac{1}{2}; \\ G_{f, \frac{1}{2}}, & p = \frac{1}{2}. \end{cases} \tag{2.5}$$

Theorem 7. *Let $f(x,y)$ be a positive n-order homogenous function defined on $\mathbb{U}(\subseteqq \mathbb{R}_+ \times \mathbb{R}_+)$, and be 3-time differentiable, and satisfies $f(x,y) = f(y,x)$ further. Let $\bar{\mathcal{H}}_{1f}(p) = \mathcal{H}_{1f}(p)\mathcal{H}_{1f}(-p)$, then the function is strictly increasing (decreasing) in $p \in (0, +\infty)$ and strictly decreasing (increasing) in $p \in (-\infty, 0)$ if $J = (x-y)(xI_1)_x < (>)0$.*

3. Lemmas

For proving Theorem 5-7 and Corollary 1, we need to the following lemmas, in which Lemma 1 and 2 are from section 3 in [14].

Lemma 1. *Let $f(x,y), g(x,y)$ be a n, m-order homogenous functions over Ω respectively, then $f \cdot g, f/g (g \neq 0)$ are $n+m, n-m$-order homogenous functions over Ω, respectively.*

If for a certain p and $(x^p, y^p) \in \Omega, f^p(x,y)$ exist, then $f(x^p, y^p), f^p(x,y)$ are both np-order homogeneous functions over Ω.

Lemma 2. *Let $f(x,y)$ be a n-order homogeneous function over Ω, and f_x, f_y both exist, then f_x, f_y are both $n-1$-order homogeneous function over Ω, furthermore we have*

$$x f_x + y f_y = n f. \tag{3.1}$$

In particular, when $n = 1$ and $f(x, y)$ is 1st differentiable over Ω, then

$$xf_x + yf_y = f; \tag{3.2}$$
$$xf_{xx} + yf_{xy} = 0; \tag{3.3}$$
$$xf_{xy} + yf_{yy} = 0. \tag{3.4}$$

Lemma 3. *Let $f(x, y)$ be a positive n-order homogenous function defined on $U(\subseteqq \mathbb{R}_+ \times \mathbb{R}_+)$, and be 3-time differentiable. Let $T(t) = \ln f(a^t, b^t)$, with $t \neq 0$, and set $a^t = x, b^t = y$, then*

$$T'(t) = \tfrac{a^t f_x(a^t,b^t)\ln a + b^t f_y(a^t,b^t)\ln b}{f(a^t,b^t)} = \ln G_f^{\frac{1}{t}}(a^t, b^t); \tag{3.5}$$
$$T''(t) = -xyI_1 \ln^2(b/a), \quad I_1 = (\ln f)_{xy}; \tag{3.6}$$
$$T'''(t) = -Ct^{-3}J, \quad J = (x - y)(xI_1)_x, \quad C = \tfrac{xy\ln^3(x/y)}{x-y} > 0. \tag{3.7}$$

Proof. 1) By a direct calculation, we obtain this result at once.

2) Since $f(x, y)$ is a positive n-order homogeneous function, from equation (3.1), we can obtain

$$x(\ln f)_x + y(\ln f)_y = n \quad \text{or} \quad x(\ln f)_x = n - y(\ln f)_y. \tag{3.8}$$

By (1.15), there is

$$\begin{aligned} T'(t) &= \frac{a^t f_x(a^t, b^t)\ln a + b^t f_y(a^t, b^t)\ln b}{f(a^t, b^t)} \\ &= \frac{xf_x(x, y)\ln a + yf_y(x, y)\ln b}{f(x, y)} \\ &= x(\ln f)_x \ln a + y(\ln f)_y \ln b \\ &= n\ln a + y(\ln f)_y(\ln b - \ln a). \end{aligned} \tag{3.9}$$

Notice that $y(\ln f)_y$ is a 0-order homogeneous function, so

$$x[y(\ln f)_y]_x + y[y(\ln f)_y]_y = 0, \quad \text{or} \quad y[y(\ln f)_y]_y = -x[y(\ln f)_y]_x \; . \tag{3.10}$$

Hence

$$\begin{aligned} T''(t) &= 0 + (\ln b - \ln a)\left[\frac{\partial y(\ln f)_y}{\partial x}\frac{dx}{dt} + \frac{\partial y(\ln f)_y}{\partial y}\frac{dy}{dt}\right] \\ &= (\ln b - \ln a)\left\{[y(\ln f)_y]_x a^t \ln a + y[y(\ln f)_y]_y b^t \ln b\right\} \\ &= \{(\ln b - \ln a)x[y(\ln f)_y]_x \ln a - x[y(\ln f)_y]_x \ln b\} \\ &= -(\ln b - \ln a)^2 x[y(\ln f)_y]_x \\ &= -xy(\ln f)_{xy}(\ln b - \ln a)^2 \\ &= -xyI_1(\ln b - \ln a)^2. \end{aligned}$$

3) From Lemma 1 and 2, we can understand that $I_1 = (\ln f)_{xy} = (ff_{xy} - f_x f_y)/f^2$ is a -2-order homogeneous function of x and y, thus xyI_1 is a 0-order homogeneous function. By (3.1), we get

$$x(xyI_1)_x + y(xyI_1)_y = 0, \text{ or } \quad y(xyI_1)_y = -x(xyI_1)_x. \tag{3.11}$$

By (3.6) and notice $x = a^t, y = b^t$, and then

$$\begin{aligned}
T'''(t) &= \frac{dT''(t)}{dt} = \frac{d\left(-xyI_1(\ln b - \ln a)^2\right)}{dt} \\
&= -(\ln b - \ln a)^2 \left[\frac{\partial(xyI_1)}{\partial x}\frac{dx}{dt} + \frac{\partial(xyI_1)}{\partial y}\frac{dy}{dt}\right] \\
&= -(\ln b - \ln a)^2 \left[a^t \ln a \cdot (xyI_1)_x + b^t \ln b \cdot (xyI_1)_y\right] \\
&= -(\ln b - \ln a)^2 \left[(x(xyI_1)_x \ln a + y \ln b (xyI_1)_y \ln b)\right] \\
&= -(\ln b - \ln a)^2 \left(x(xyI_1)_x\right)(\ln a - \ln b) \\
&= (\ln b - \ln a)^3 xy(xI_1)_x \\
&= xy\frac{(\ln b - \ln a)^3}{x-y}\left[(x-y)(xI_1)_x\right] \\
&= -xy\frac{(\ln x - \ln y)^3}{t^3(x-y)}\left[(x-y)(xI_1)_x\right] \\
&= -Ct^{-3}J.
\end{aligned}$$

∎

Remark 1. *By Lemma 3, it is not difficult to get the following conclusions:*

1) $T(t)$ is strictly convex (concave) in $t \in (-\infty, 0) \cup (0, +\infty)$ if $I_1 < (>)0$;

2) $T'(t)$ is strictly increasing (decreasing) in $t \in (-\infty, 0) \cup (0, +\infty)$ if $I_1 < (>)0$;

3) If $J < (>)0$, then $T'(t)$ is strictly convex (concave) in $t \in (0, +\infty)$, and strictly concave (convex) in $t \in (-\infty, 0)$.

4) If $J < (>)0$, then $T''(t)$ is strictly increasing (decreasing) in $t \in (0, +\infty)$, and strictly decreasing (increasing) in $t \in (-\infty, 0)$.

Lemma 4. *The conditions of this Lemma are the same as Lemma 3, and $f(x, y)$ is symmetric with respect to x and y, then the following equations hold:*

$$T'(t) + T'(-t) = 2n \ln G, \tag{3.12}$$

$$T''(-t) = T''(t). \tag{3.13}$$

$$T'''(-t) = -T'''(t). \tag{3.14}$$

Proof. By direct calculations of the first, second and third derivative to variable t in two sides of equation (1.17) respectively, the equations (3.12)-(3.14) are derived immediately. The proof is completed. ∎

Remark 2. *If $(1, 1) \in U$, i.e. $T'(0)$ exists, then $T'(0) = n \ln G$; If $(1, 1) \notin U$, we define $T'(0) = \lim_{t\to 0} T'(t) = n \ln G$. Thus the (3.12) can be written as*

$$T'(t) + T'(-t) = 2T'(0). \tag{3.15}$$

4. Proofs of the main results

Applying the Lemmas 1-4, we can prove the theorems and corollary in section 2.

*proof of Theorem **5**.*

$$\ln \mathcal{H}_{1f}(p) = \ln \frac{f(a^{p+1}, b^{p+1})}{f(a^p, b^p)} = T(p+1) - T(p) \tag{4.1}$$

$$\frac{\mathrm{d} \ln \mathcal{H}_{1f}(p)}{\mathrm{d}p} = T'(p+1) - T'(p) \tag{4.2}$$

From Lemma 3, we see that $T'(t)$ is strictly increasing (decreasing) in $t \in (-\infty, 0) \cup (0, +\infty)$ if $I_1 < (>)0$, so $T'(p+1) - T'(p) > (<)0$ for $p > 0$ or $p < -1$; For $-1 < p < 0$, we have

$$T'(p+1) > (<)T'(0) > (<)T'(p), \quad \text{i.e.} \quad T'(p+1) - T'(p) > (<)0.$$

It shows that $\mathcal{H}_{1f}(p)$ is strictly increasing (decreasing) in $p \in (-\infty, 0) \cup (0.+\infty)$ if $I < (>)0$. it follows this theorem. ∎

proof of Theorem 6. 1)By the process of proof of Theorem 5, we see that

$$\frac{\mathrm{d}^2 \ln \mathcal{H}_{1f}(p)}{\mathrm{d}p^2} = T''(p+1) - T''(p). \tag{4.3}$$

Since $T'''(t) = -CJ/t^3$, so $T''(t)$ is strictly increasing in $t \in (0, +\infty)$ if $J < 0$, strictly decreasing in $t \in (-\infty, 0)$. And then $T''(p+1) - T''(p) > 0$ if $p > 0$, and $T''(p+1) - T''(p) < 0$ if $p < -1$. In other words, $\ln\mathcal{H}_{1f}(p)$ is convex on $(0, +\infty)$, concave on $(-\infty, -1)$.

For $J = (x-y)(xI)_x > 0$, clearly, the above conclusion is reversed.

2) From part 1), the convexity of $\ln\mathcal{H}_{1f}(p)$ on $(-\infty, -1)$ or $(0, +\infty)$ has been confirmed, and needs to verify on $p \in (-1, 0)$ further.

By Lemma 4, there is $T''(-p) = T''(p)$ if $f(x, y) = f(y, x)$, so

$$\frac{\mathrm{d}^2 \ln \mathcal{H}_{1f}(p)}{\mathrm{d}p^2} = T''(p+1) - T''(p) = T''(p+1) - T''(-p). \tag{4.4}$$

If $J = (x-y)(xI)_x < 0$, then $T''(p+1) - T''(-p) > 0$ in $p \in (-\frac{1}{2}, 0)$, and $T''(p+1) - T''(-p) < 0$ in $p \in (-1, -\frac{1}{2})$. Namely, $\ln\mathcal{H}_{1f}(p)$ is convex on $(-\frac{1}{2}, 0)$, concave on $(-1, -\frac{1}{2})$.

Combining 1) with 2), the proof is completed. ∎

Proof of Theorem 7. Since $\bar{\mathcal{H}}_{1f}(p) = \mathcal{H}_{1f}(p)\mathcal{H}_{1f}(-p)$, so we have

$$\ln \bar{\mathcal{H}}_{1f}(p) = T(p+1) - T(p) + T(-p+1) - T(-p), \tag{4.5}$$

$$\frac{\mathrm{d} \ln \bar{\mathcal{H}}_{1f}(p)}{\mathrm{d}p} = T'(p+1) - T'(p) - T'(-p+1) + T'(-p). \tag{4.6}$$

By Lemma 4, (4.6) can be written as

$$\frac{\mathrm{d} \ln \bar{\mathcal{H}}_{1f}(p)}{\mathrm{d}p} = \begin{cases} T'(p+1) + T'(p-1) - 2T'(p), & p \in [1, +\infty); \\ T'(p+1) - T'(1-p) - 2[T'(p) - T'(0)], & p \in (0, 1). \end{cases} \tag{4.7}$$

if $J = (x-y)(xI)_x < 0$, then $T'''(t) > (<)0$ when $t > (<)0$, i.e. that $T'(t)$ is strictly convex (concave) in $t > (<)0$. By the properties of convex (concave), we easily get

$$\frac{T'(p+1) + T'(p-1)}{2} > T'(p) \text{ if } p \in (1, +\infty); \tag{4.8}$$

While for $p \in (0,1)$, because

$$\frac{T'(p+1)-T'(1-p)}{(p+1)-(1-p)} > \frac{T'(p)-T'(1-p)}{p-(1-p)} > \frac{T'(p)-T'(0)}{p-0}, \tag{4.9}$$

so there is

$$T'(p+1)-T'(1-p) > 2[T'(p)-T'(0)]. \tag{4.10}$$

It follows that whether $p \in [1,+\infty)$ or $p \in (0,1)$ there are always $\frac{\mathrm{d}\ln \bar{\mathcal{H}}_{1f}(p)}{\mathrm{d}p} > 0$, i.e. $\bar{\mathcal{H}}_{1f}(p)$ is strictly increasing in $p \in (0,+\infty)$ if $J<0$.

As $\bar{\mathcal{H}}_{1f}(-p) = \mathcal{H}_{1f}(-p)\mathcal{H}_{1f}(p) = \bar{\mathcal{H}}_{1f}(p)$, so $\bar{\mathcal{H}}_{1f}(p)$ is strictly decreasing in $p \in (-\infty,0)$ at the same time.

For $J=(x-y)(xI)_x>0$, we can prove the conclusion in the same way. ∎

5. Some conclusions involving L, A and E

By Theorem 5-7, the monotonicity of $\mathcal{H}_{1f}(p)$ depends on the sign of $I_1 = (\ln f)_{xy}$; While the log-convexity of $\mathcal{H}_{1f}(p)$, the monotonicity of $\mathcal{H}_f(p,1-p)$ and $\bar{\mathcal{H}}_{1f}(p)$ depend on the sign of $J=(x-y)(xI_1)_x$. In this section, by some straightforward computations, we will present some conclusions about $\mathcal{H}_{1f}(p)$,$\mathcal{H}_f(p,1-p)$ and $\bar{\mathcal{H}}_{1f}(p)$, where $f(x,y)=L(x,y),A(x,y),E(x,y)$.

Case 1. *For $f(x,y)=L(x,y)=\dfrac{x-y}{\ln x-\ln y}$, where $x,y>0$ with $x\neq y$, there are*

$$\begin{aligned}
I_1 &= (\ln f)_{xy} = \frac{1}{(x-y)^2} - \frac{1}{xy(\ln x-\ln y)^2}\\
&= \frac{1}{xy(x-y)^2}[G^2(x,y)-L^2(x,y),\\
J &= (x-y)(xI_1)_x = (x-y)\left[-\frac{x+y}{(x-y)^3}+\frac{2}{xy(\ln x-\ln y)^3}\right]\\
&= \frac{2}{xy(x-y)^2}\left[L^3(x,y)-\frac{x+y}{2}\left(\sqrt{xy}\right)^2\right].
\end{aligned}$$

By the well-known inequalities $L(x,y)>G(x,y)$ and $L(x,y)>\left(\frac{x+y}{2}\right)^{\frac{1}{3}}\left(\sqrt{xy}\right)^{\frac{2}{3}}$, we have $I_1<0, J>0$.

Case 2. *For $f(x,y)=A(x,y)=\dfrac{x+y}{2}$, where $x,y>0$ with $x\neq y$, there are*

$$\begin{aligned}
I_1 &= (\ln f)_{xy} = -\frac{1}{(x+y)^2}<0,\\
J &= (x-y)(xI_1)_x = \frac{(x-y)^2}{(x+y)^3}>0.
\end{aligned}$$

Case 3. *For* $f(x,y)=E(x,y)=e^{-1}\left(\frac{x^x}{y^y}\right)^{\frac{1}{x-y}}$*, where* $x,y>0$ *with* $x\neq y$*, there are*

$$\begin{aligned} I_1 &= (\ln f)_{xy}=\frac{1}{(x-y)^3}\left[2(x-y)-(x+y)(\ln x-\ln y)\right] \\ &= \frac{2(\ln x-\ln y)}{(x-y)^3}\left[L(x,y)-\frac{x+y}{2}\right] \\ J &= (x-y)(xI_1)_x=\frac{-3(x^2-y^2)+(x^2+4xy+y^2)(\ln x-\ln y)}{(x-y)^2} \\ &= -\frac{6(\ln x-\ln y)}{(x-y)^3}\left[\frac{x^2-y^2}{\ln x^2-\ln y^2}-\frac{\frac{x^2+y^2}{2}+2xy}{3}\right]. \end{aligned}$$

By the well-known inequalities $L(x,y)<\frac{x+y}{2}$ *and* $L(x,y)<\frac{\frac{x+y}{2}+2\sqrt{xy}}{3}$*, we have* $I_1<0, J>0$*.*

Case 4. *For* $f(x,y)=D(x,y)=|x-y|$*, where* $x,y>0$ *with* $x\neq y$*, there are*

$$\begin{aligned} I_1 &= (\ln f)_{xy}=\frac{1}{(x-y)^2}>0 \\ J &= (x-y)(xI_1)_x=-\frac{x+y}{(x-y)^2}<0 \end{aligned}$$

Notice that $L(x,y), A(x,y), E(x,y)$ and $D(x,y)$ are all symmetric with respect to x and y, using Theorems 5-7 and Corollary 1, we get immediately the following conclusions:

Conclusion 1. *That* $\mathcal{H}_{1L}(a,b;p)$,$\mathcal{H}_{1A}(a,b;p)$ *and* $\mathcal{H}_{1E}(a,b;p)$ *are strictly increasing in* $p\in(-\infty,+\infty)$*, respectively.*

That $\mathcal{H}_{1D}(a,b;p)$ *is strictly decreasing in* $p\in(-\infty,0)\cup(0,+\infty)$*.*

Conclusion 2. *That* $\mathcal{H}_{1L}(a,b;p)$,$\mathcal{H}_{1A}(a,b;p)$ *and* $\mathcal{H}_{1E}(a,b;p)$ *are strictly log-convex in* $p\in(-\infty,-\frac{1}{2})$*, and strictly log-concave in* $p\in(-\frac{1}{2},+\infty)$*, respectively.*

That $\mathcal{H}_{1D}(a,b;p)$ *is strictly log-concave in* $p\in(-\infty,-\frac{1}{2})$*, and strictly log-convex in* $p\in(-\frac{1}{2},0)\cup(0,+\infty)$*.*

Conclusion 3. *That* $\mathcal{H}_{1L}(p,1-p)$,$\mathcal{H}_{1A}(p,1-p)$ *and* $\mathcal{H}_{1E}(p,1-p)$ *are strictly increasing in* $p\in(-\infty,\frac{1}{2})$*, and strictly decreasing in* $p\in(\frac{1}{2},+\infty)$*, respectively.*

That $\mathcal{H}_{1D}(p,1-p)$ *is strictly decreasing in* $p\in(-\infty,0)\cup(0,\frac{1}{2})$*, and strictly increasing in* $p\in(\frac{1}{2},+\infty)$*.*

Conclusion 4. *That* $\bar{\mathcal{H}}_{1L}(a,b;p)$, $\bar{\mathcal{H}}_{1A}(a,b;p)$ *and*
$\bar{\mathcal{H}}_{1E}(a,b;p)$ *are strictly increasing in* $p\in(-\infty,0)$*, and strictly decreasing in* $p\in(0,+\infty)$*, respectively.*

That $\bar{\mathcal{H}}_{1D}(a,b;p)$ *is strictly decreasing in* $p\in(-\infty,0)$*, and strictly increasing in* $p\in(0,+\infty)$*,*

Remark 3. *The Conclusion 2 and 4 include Wing-Sum Cheung and Feng Qi's results.*

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