BOUNDS FOR THE ČEBYŠEV FUNCTIONAL OF A CONVEX AND A BOUNDED FUNCTION

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Abstract. Upper and lower bounds for the Čebyšev functional of a convex and a bounded function are given. Some applications for quadrature rules and probability density functionsare also provided.

1. Introduction

For two Lebesgue functions $f, g : [a, b] \to \mathbb{R}$, consider the Čebyšev functional

$$C(f, g) := \frac{1}{b-a} \int_a^b f(t) g(t) \, dt - \frac{1}{b-a} \int_a^b f(t) \, dt \cdot \frac{1}{b-a} \int_a^b g(t) \, dt. \tag{1.1}$$

In 1971, F.V. Atkinson [1] showed that if f, g are twice differentiable and convex on $[a, b]$ and

$$\int_a^b \left(t - \frac{a+b}{2} \right) g(t) \, dt = 0, \tag{1.2}$$

then $C(f, g)$ is nonnegative.

This result is, in fact, implied by that of A. Lupaş [3] who proved that for any two convex functions $f, g : [a, b] \to \mathbb{R}$ the lower bound for the Čebyšev functional is:

$$C(f, g) \geq \frac{12}{(b-a)^3} \int_a^b \left(t - \frac{a+b}{2} \right) f(t) \, dt \cdot \int_a^b \left(t - \frac{a+b}{2} \right) g(t) \, dt, \tag{1.3}$$

with true equality holding when at least one of f or g is a linear function on $[a, b]$.

As pointed out in [4, p. 262], if the functions f, g are convex and one is symmetric, then $C(f, g) \geq 0$.

For other results for convex integrands, see [4, p. 256] and [4, p. 262] where further references are given.

In this note we provide some bounds for the Čebyšev functional in the case of a convex function g and a bounded function f. Some applications are given as well.

2. The Results

For an integrable function $f : [a, b] \to \mathbb{R}$, define the $(\gamma - 2)$ −moment by

$$M_{2,\gamma}(f) := \int_a^b (t - \gamma)^2 f(t) \, dt.$$

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For a convex function $g:[a,b]\to\mathbb{R}$ for which the derivatives $g_{-}'(b)$ and $g_{+}'(a)$ are finite, define

$$\Gamma(f,g):=\frac{g_{-}'(b)\,M_{2,b}(f)-g_{+}'(a)\,M_{2,a}(f)}{2(b-a)^{2}},$$

where f is integrable on $[a,b]$.

The following result holds:

Theorem 1. *If $f:[a,b]\to\mathbb{R}$ is a Lebesgue measurable function such that there exists the constants $m,M\in\mathbb{R}$ with*

$$m\leq f(t)\leq M\quad \text{for a.e.}\quad t\in[a,b], \tag{2.1}$$

and $g:[a,b]\to\mathbb{R}$ is a convex function on $[a,b]$ with the lateral derivatives $g_{+}'(a)$ and $g_{-}'(b)$ finite, then,

$$\begin{aligned}&\frac{1}{6}m(b-a)\left[g_{-}'(b)-g_{+}'(a)\right]-\Gamma(f,g)\\&\leq C(f,g)\\&\leq\frac{1}{6}M(b-a)\left[g_{-}'(b)-g_{+}'(a)\right]-\Gamma(f,g).\end{aligned}\tag{2.2}$$

Proof. We use Sonin's identity [4, p. 246]:

$$C(f,g)=\frac{1}{b-a}\int_{a}^{b}(f(t)-\gamma)\left(g(t)-\frac{1}{b-a}\int_{a}^{b}g(s)\,ds\right)dt, \tag{2.3}$$

for any $\gamma\in\mathbb{R}$, and the following inequality for convex functions obtained by S.S. Dragomir in [2]:

$$\frac{1}{b-a}\int_{a}^{b}g(s)\,ds-g(t)\leq\frac{1}{2(b-a)}\left[(b-t)^{2}g_{-}'(b)-(t-a)^{2}g_{+}'(a)\right] \tag{2.4}$$

for any $t\in[a,b]$. The constant $\frac{1}{2}$ is sharp.

Now, by Sonin's identity for $\gamma=M$, we have

$$C(f,g)=\frac{1}{b-a}\int_{a}^{b}(M-f(t))\left(\frac{1}{b-a}\int_{a}^{b}g(s)\,ds-g(t)\right)dt. \tag{2.5}$$

From (2.4) we get

$$\begin{aligned}&\left(\frac{1}{b-a}\int_{a}^{b}g(s)\,ds-g(t)\right)(M-f(t))\\&\qquad\leq\frac{1}{2(b-a)}\left[g_{-}'(b)(b-t)^{2}(M-f(t))-g_{+}'(a)(t-a)^{2}(M-f(t))\right]\end{aligned}\tag{2.6}$$

for a.e. $t\in[a,b]$.

Integrating (2.6) over t on $[a,b]$ and using the representation (2.5), we get

$$\begin{aligned}C(f,g)\leq\frac{1}{2(b-a)^{2}}\Bigg[&M\int_{a}^{b}\left[g_{-}'(b)(b-t)^{2}-g_{+}'(a)(t-a)^{2}\right]dt\\&-g_{-}'(b)\int_{a}^{b}(b-t)^{2}f(t)\,dt+g_{+}'(a)\int_{a}^{b}(t-a)^{2}f(t)\,dt\Bigg].\end{aligned}\tag{2.7}$$

Since

$$\int_a^b \left[g'_-(b)(b-t)^2 - g'_+(a)(t-a)^2\right]dt = \frac{(b-a)^3}{3}\left[g'_-(b) - g'_+(a)\right]$$

then (2.7) provides the second part of (2.2).

Again, by Sonin's identity,

$$C(f,g) = \frac{1}{b-a}\int_a^b (m - f(t))\left(\frac{1}{b-a}\int_a^b g(s)\,ds - g(t)\right)dt.$$

Utilising (2.4) and the fact that $m - f(t) \leq 0$ for a.e. $t \in [a,b]$, we obtain,

$$\begin{aligned} &C(f,g) \\ &\geq \frac{1}{2(b-a)^2}\int_a^b \left[(b-t)^2 g'_-(b)(m-f(t)) - (t-a)^2 g'_+(a)(m-f(t))\right]dt \\ &= \frac{1}{2(b-a)^2}\left[m\int_a^b \left[(b-t)^2 g'_-(b) - (t-a)^2 g'_+(a)\right]dt - 2(b-a)\Gamma(f,g)\right], \end{aligned}$$

giving the first part of (2.2). ∎

The following particular result holds.

Corollary 1. *Let $f : [a,b] \to \mathbb{R}$ be a Lebesgue measurable essentially bounded function on $[a,b]$, i.e., $f \in L_\infty[a,b]$ and $\|f\|_\infty := ess\sup_{t\in[a,b]}|f(t)|$ its norm. If $g : [a,b] \to \mathbb{R}$ is a convex function on $[a,b]$ with the lateral derivatives $g'_+(a)$ and $g'_-(b)$ finite, then we have the inequality:*

$$|C(f,g) + \Gamma(f,g)| \leq \frac{1}{6}\|f\|_\infty (b-a)\left[g'_-(b) - g'_+(a)\right]. \tag{2.8}$$

3. Applications for the Trapezoid Rule

The following result is a perturbed version of the trapezoid rule.

Proposition 1. *Let $h : [a,b] \to \mathbb{R}$ be a differentiable function with the property that the derivative $h' : (a,b) \to \mathbb{R}$ is convex on (a,b). If $h''_+(a)$, $h''_-(b)$ are finite, then*

$$\begin{aligned} &\left|\frac{h(a)+h(b)}{2} - \frac{1}{b-a}\int_a^b h(t)\,dt - \frac{(b-a)^2}{12}\cdot\frac{h''_+(a)+h''_-(b)}{2}\right| \\ &\qquad\qquad \leq \frac{1}{24}(b-a)^2\cdot\left[h''_-(b) - h''_+(a)\right]. \end{aligned} \tag{3.1}$$

Proof. Consider the functions $f, g : [a,b] \to \mathbb{R}$ defined by

$$f(t) = t - \frac{a+b}{2},\; g(t) = h'(t).$$

For these functions, a simple calculation shows that

$$\Gamma(f,g) = \frac{(b-a)^2}{12}\cdot\frac{h''_+(a)+h''_-(b)}{2},$$

since,

$$\int_a^b (t-b)^2\left(t - \frac{a+b}{2}\right)dt = -\frac{(b-a)^4}{12}$$

and

$$\int_a^b (t-a)^2 \left(t - \frac{a+b}{2}\right) dt = \frac{(b-a)^4}{12}.$$

Clearly, also,

$$\|f\|_\infty = \frac{1}{2}(b-a).$$

Utilising the elementary identity

$$\frac{1}{b-a}\int_a^b \left(t-\frac{a+b}{2}\right) h'(t)\, dt = \frac{h(a)+h(b)}{2} - \frac{1}{b-a}\int_a^b h(t)\, dt$$

and the fact that, for f, g as defined previously

$$C(f,g) = \frac{1}{b-a}\int_a^b \left(t-\frac{a+b}{2}\right) h'(t)\, dt,$$

a direct application of Corollary 1 reveals the desired inequality (3.1). ∎

A second result in the same spirit may be stated as:

Proposition 2. *Let* $h : [a,b] \to \mathbb{R}$ *be a twice differentiable function with the property that the second derivative* $h'' : (a,b) \to \mathbb{R}$ *is convex on* (a,b). *If* $h'''_+(a)$, $h'''_-(b)$ *are finite, then*

$$\left|\frac{h(a)+h(b)}{2} - \frac{1}{b-a}\int_a^b h(t)\, dt + \frac{b-a}{12}\cdot\left[h'(b)-h'(a)\right] - \frac{1}{80}\left[h'''_-(b)-h'''_+(a)\right](b-a)^3\right| \leq \frac{1}{48}(b-a)^3 \cdot \left[h'''_-(b)-h'''_+(a)\right]. \tag{3.2}$$

Proof. Consider the functions $f, g : [a,b] \to \mathbb{R}$ defined by

$$f(t) = \frac{1}{2}(t-a)(t-b),\ g(t) = h''(t).$$

A simple calculation shows that,

$$\Gamma(f,g) = -\frac{1}{80}(b-a)^3 \cdot \left[h'''_-(b) - h'''_+(a)\right],$$

since,

$$\frac{1}{2}\int_a^b (t-b)^2 (t-a)(t-b)\, dt = -\frac{(b-a)^5}{40}$$

and

$$\frac{1}{2}\int_a^b (t-a)^2 (t-a)(t-b)\, dt = -\frac{(b-a)^5}{40}.$$

It can also be seen that,

$$\|f\|_\infty = \frac{1}{8}(b-a)^2.$$

Utilising the elementary identity

$$\frac{1}{b-a}\int_a^b \left[\frac{1}{2}(t-a)(t-b)\right] h''(t)\, dt = \frac{1}{b-a}\int_a^b h(t)\, dt - \frac{h(a)+h(b)}{2}$$

and the fact that, for f, g as defined previously,

$$C(f,g) = \frac{1}{b-a}\int_a^b \left[\frac{1}{2}(t-a)(t-b)\right] h''(t)\,dt + \frac{b-a}{12}\cdot\left[h'(b) - h'(a)\right],$$

a direct application of Corollary 1 reveals the desired inequality (3.1). ∎

Remark 1. *Similar results may be stated if one considers quadrature rules for which the remainder $R(f)$ can be expressed in Peano kernel form, i.e.,*

$$R(f) = \int_a^b K(t)\, f^{(n)}(t)\,dt$$

where $K(t)$ is a kernel for which the supremum norm can be easily computed and the $n-th$ derivative of the function f is assumed to be convex on (a,b). The exploration of these bounds is left to the interested reader.

4. Applications for Probability Density Functions

Let $f : [a,b] \to [0,\infty)$ be a *density function*, this means that f is integrable on $[a,b]$ and $\int_a^b f(t)\,dt = 1$ and let

$$F(x) := \int_a^x f(t)\,dt, \quad x \in [a,b]$$

be its *distribution function.* We also denote the *expectation* of f by $E(f)$, where

$$E(f) := \int_a^b t f(t)\,dt,$$

provided that the integral exists and is finite, and the *mean deviation* $M_D(f)$, by

$$M_D(f) := \int_a^b |t - E(f)|\, f(t)\,dt.$$

Theorem 2. *Let $f : [a,b] \to [0,\infty)$ be a density function with the property that there exists $m, M \geq 0$ such that*

$$m \leq f(t) \leq M \quad \text{for a.e.} \quad t \in [a,b]$$

then

$$\frac{1}{3}m(b-a)^2 \leq M_D(f) + \frac{1}{b-a}M_{2,\frac{a+b}{2}}(f) - \frac{\left(E(f) - \frac{a+b}{2}\right)^2}{b-a} \leq \frac{1}{3}M(b-a)^2. \tag{4.1}$$

Proof. We apply Theorem 1 for $g : [a,b] \to \mathbb{R}$, $g(t) = |t - E(f)|$. Since

$$g'_-(b) = 1, \qquad g'_+(a) = -1,$$

then

$$\begin{aligned}\Gamma(f,g) &= \frac{1}{(b-a)^2}\int_a^b \left[\frac{(t-a)^2 + (t-b)^2}{2}\right] f(t)\,dt \\ &= \frac{1}{(b-a)^2}\int_a^b \left[\left(t - \frac{a+b}{2}\right)^2 + \frac{1}{4}(b-a)^2\right] f(t)\,dt \\ &= \frac{1}{(b-a)^2}M_{2,\frac{a+b}{2}}(f) + \frac{1}{4}.\end{aligned}$$

On the other hand,

$$\begin{aligned} C(f,g) &= \frac{1}{b-a}\int_a^b |t-E(f)|\, f(t)\,dt - \frac{1}{b-a}\int_a^b |t-E(f)|\,dt \cdot \frac{1}{b-a}\int_a^b f(t)\,dt \\ &= \frac{1}{b-a}M_D(f) - \frac{1}{(b-a)^2}\left[\frac{(b-E(f))^2+(E(f)-a)^2}{2}\right] \\ &= \frac{1}{b-a}M_D(f) - \frac{1}{(b-a)^2}\left[\left(E(f)-\frac{a+b}{2}\right)^2+\frac{1}{4}(b-a)^2\right] \\ &= \frac{1}{b-a}M_D(f) - \frac{\left(E(f)-\frac{a+b}{2}\right)^2}{(b-a)^2} - \frac{1}{4}. \end{aligned}$$

Making use of the inequality (2.2) we deduce the desired result (4.1). ∎

If one is interested in providing bounds for the *absolute moment* around the midpoint $\frac{a+b}{2}$,

$$M_{\frac{a+b}{2}}(f) := \int_a^b \left|t-\frac{a+b}{2}\right| f(t)\,dt,$$

then on applying Theorem 1 for $g(t)=\left|t-\frac{a+b}{2}\right|$, we have the following

Theorem 3. *Let* $f:[a,b]\to[0,\infty)$ *be as in Theorem 2. Then*

$$\frac{1}{3}m(b-a)^2 \le M_{\frac{a+b}{2}}(f) + \frac{1}{b-a}M_{2,\frac{a+b}{2}}(f) \le \frac{1}{3}M(b-a)^2. \tag{4.2}$$

Remark 2. *Similar results may be stated if one considers higher moments*

$$M_{p,\gamma}(f) := \int_a^b |t-\gamma|^p f(t)\,dt, \qquad p\ge 1,$$

for which $g(t)=|t-\gamma|^p$ *in Theorem 1 will procure the corresponding bounds in terms of* m *and* M *with the property that* $0<m\le f(t)\le M$ *for a.e.* $t\in[a,b]$. *The details are omitted.*

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