BESSEL TYPE INEQUALITIES FOR NON-ORTHONORMAL FAMILIES OF VECTORS IN INNER PRODUCT SPACES

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Abstract. Some sharp Bessel type inequalities for non-orthonormal families of vectors in inner product spaces are given. Applications for complex numbers are also provided.

1. Introduction

Let $(H; \langle \cdot, \cdot \rangle)$ be an inner product space over the real or complex number field $\mathbb{K}$. If $\{e_i\}_{i \in \{1,\dots,n\}}$ are orthonormal vectors in H, i.e., $\langle e_i, e_j \rangle = \delta_{ij}$ for all $i, j \in \{1, \dots, n\}$, where δ_{ij} is the Kronecker delta, then we have the following inequality:

$$\sum_{j=1}^{n} |\langle x, e_j \rangle|^2 \le \|x\|^2 \quad \text{for any} \quad x \in H, \tag{1.1}$$

which is well known in the literature as Bessel's inequality.

In 1941, R.P. Boas [2] and in 1944, independently, R. Bellman [1] proved the following generalisation of Bessel's inequality (see also [8, p. 392]):

Theorem 1 (Boas-Bellman; 1941, 1944)**.** *If $x, y_1, \dots, y_n$ are vectors in an inner product space $(H; \langle \cdot, \cdot \rangle)$, then*

$$\sum_{i=1}^{n} |\langle x, y_i \rangle|^2 \le \|x\|^2 \left[\max_{1 \le i \le n} \|y_i\|^2 + \left(\sum_{1 \le i \ne j \le n} |\langle y_i, y_j \rangle|^2 \right)^{\frac{1}{2}} \right]. \tag{1.2}$$

In 1971, E. Bombieri [3] (see also [8, p. 394]) gave the following generalisation of Bessel's inequality:

Theorem 2 (Bombieri, 1971)**.** *Let $x, y_1, \dots, y_n$ be vectors in $(H; \langle \cdot, \cdot \rangle)$. Then*

$$\sum_{i=1}^{n} |\langle x, y_i \rangle|^2 \le \|x\|^2 \max_{1 \le i \le n} \left\{ \sum_{j=1}^{n} |\langle y_i, y_j \rangle| \right\}. \tag{1.3}$$

Another generalisation of Bessel's inequality was obtained by A. Selberg (see for example [8, p. 394]):

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Theorem 3 (Selberg)**.** *Let $x, y_1, \dots, y_n$ be vectors in H with $y_i \neq 0$ for $i \in \{1, \dots, n\}$. Then*

$$\sum_{i=1}^{n} \frac{|\langle x, y_i \rangle|^2}{\sum_{j=1}^{n} |\langle y_i, y_j \rangle|} \leq \|x\|^2 . \tag{1.4}$$

In 2003, the author obtained the following Boas-Bellman type inequality [4]:

Theorem 4 (Dragomir, 2003)**.** *For any $x, y_1, \dots, y_n \in H$ one has*

$$\sum_{i=1}^{n} |\langle x, y_i \rangle|^2 \leq \|x\|^2 \left\{ \max_{1 \leq i \leq n} \|y_i\|^2 + (n-1) \max_{1 \leq i \neq j \leq n} |\langle y_i, y_j \rangle| \right\} . \tag{1.5}$$

We remark that in all inequalities (1.2) – (1.5) the case when $\{y_i\}_{i \in \{1,\dots,n\}}$ is an orthonormal family produces the classical Bessel's inequality.

A generalisation of the Bombieri result for the pair (p, q) with $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$ has been obtained by the author in 2003, [5].

Theorem 5 (Dragomir, 2003)**.** *For any $x, y_1, \dots, y_n \in H$ with not all $\langle x, y_i \rangle = 0$ for $i \in \{1, \dots, n\}$, one has:*

$$\frac{\left(\sum_{i=1}^{n} |\langle x, y_i \rangle|^2 \right)^2}{\left(\sum_{i=1}^{n} |\langle x, y_i \rangle|^p \right)^{\frac{1}{p}} \left(\sum_{i=1}^{n} |\langle x, y_i \rangle|^q \right)^{\frac{1}{q}}} \leq \|x\|^2 \max_{1 \leq i \leq n} \left(\sum_{j=1}^{n} |\langle y_i, y_j \rangle| \right) . \tag{1.6}$$

If in this inequality one considers $p = q = 2$, then one obtains Bombieri's result (1.3).

From a different perspective, the following result for the sum of Fourier coefficients due to H. Heilbronn may be stated [7] (see also [8, p. 395]).

Theorem 6 (Heilbronn, 1958)**.** *For any $x, y_1, \dots, y_n \in H$ one has*

$$\sum_{i=1}^{n} |\langle x, y_i \rangle| \leq \|x\| \left(\sum_{i,j=1}^{n} |\langle y_i, y_j \rangle| \right)^{\frac{1}{2}} . \tag{1.7}$$

In 1992, Pečarić [9] (see also [8, p. 394]) proved the following inequality that incorporates some of the results above:

Theorem 7 (Pečarić, 1992)**.** *Let $x, y_1, \dots, y_n \in H$ and $c_1, \dots, c_n \in \mathbb{K}$. Then*

$$\begin{aligned} \left| \sum_{k=1}^{n} c_k \langle x, y_k \rangle \right|^2 &\leq \|x\|^2 \sum_{i=1}^{n} |c_i|^2 \left(\sum_{j=1}^{n} |\langle y_i, y_j \rangle| \right) \\ &\leq \|x\|^2 \sum_{k=1}^{n} |c_k|^2 \max_{1 \leq i \leq n} \left\{ \sum_{j=1}^{n} |\langle y_i, y_j \rangle| \right\} . \end{aligned} \tag{1.8}$$

He showed that the Bombieri inequality (1.3) may be obtained from (1.8) for the choice $c_i = \overline{\langle x, y_i \rangle}$ (using the second inequality), the Selberg inequality (1.4) may be obtained from the first part of (1.8) for the choice

$$c_i = \frac{\overline{\langle x, y_i \rangle}}{\sum_{j=1}^{n} |\langle y_i, y_j \rangle|}, \quad i \in \{1, \dots, n\} ,$$

while the Heilbronn inequality (1.7) may be obtained from the first part of (1.8) if one chooses

$$c_i = \frac{\overline{\langle x, y_i\rangle}}{|\langle x, y_i\rangle|}, \quad i \in \{1, \dots, n\}.$$

In the spirit of Pečarić's result, the author proved in [5] the following result as well:

Theorem 8 (Dragomir, 2004)**.** *Let $x, y_1, \dots, y_n \in H$ and $c_1, \dots, c_n \in \mathbb{K}$. Then*

$$\left|\sum_{k=1}^{n} c_k \langle x, y_k\rangle\right|^2 \tag{1.9}$$

$$\leq \|x\|^2 \times \begin{cases} \max\limits_{1\le i\le n} |c_k| \sum\limits_{k=1}^{n} |c_k| \max\limits_{1\le i\le n} \left(\sum\limits_{j=1}^{n} |\langle y_i, y_j\rangle|\right); \\ \\ \sum\limits_{k=1}^{n} |c_k| \left(\sum\limits_{k=1}^{n} |c_k|^p\right)^{\frac{1}{p}} \max\limits_{1\le i\le n} \left\{\left(\sum\limits_{j=1}^{n} |\langle y_i, y_j\rangle|^q\right)^{\frac{1}{q}}\right\}, \\ \qquad\qquad p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \\ \left(\sum\limits_{k=1}^{n} |c_k|\right)^2 \max\limits_{1\le i,j\le n} |\langle y_i, y_j\rangle|. \end{cases}$$

In particular, for the choice $c_k = \overline{\langle x, y_k\rangle}$, $k \in \{1, \dots, n\}$, we deduce from (1.9) that

$$\frac{\left(\sum_{k=1}^{n} |\langle x, y_k\rangle|^2\right)^2}{\max\limits_{1\le k\le n} |\langle x, y_k\rangle| \sum_{k=1}^{n} |\langle x, y_k\rangle|} \le \|x\|^2 \max_{1\le i\le n}\left\{\sum_{j=1}^{n} |\langle y_i, y_j\rangle|\right\}, \tag{1.10}$$

$$\frac{\left(\sum_{k=1}^{n} |\langle x, y_k\rangle|^2\right)^2}{\sum_{k=1}^{n} |\langle x, y_k\rangle| \left(\sum_{k=1}^{n} |\langle x, y_k\rangle|^p\right)^{\frac{1}{p}}} \le \|x\|^2 \max_{1\le i\le n}\left\{\left(\sum_{j=1}^{n} |\langle y_i, y_j\rangle|^q\right)^{\frac{1}{q}}\right\}, \tag{1.11}$$

for $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$; and

$$\frac{\left(\sum_{k=1}^{n} |\langle x, y_k\rangle|^2\right)^2}{\left(\sum_{k=1}^{n} |\langle x, y_k\rangle|\right)^2} \le \|x\|^2 \max_{1\le i,j\le n} |\langle y_i, y_j\rangle|, \tag{1.12}$$

provided not all $\langle x, y_k\rangle$, $k \in \{1, \dots, n\}$ are zero.

The aim of the present paper is to provide different upper bounds for the Bessel sum $\sum_{i=1}^{n} |\langle x, y_i\rangle|^2$ under various conditions for the vectors enclosed. Applications for complex numbers are provided as well.

2. The Results

The following sharp inequality of Bessel type for Fourier coefficients satisfying some restrictions may be stated:

Theorem 9. *Let $(H;\langle\cdot,\cdot\rangle)$ be a real or complex inner product space, $\gamma,\Gamma\in\mathbb{K}$ with $\Gamma\neq-\gamma$ and $x,y_j\in H$, $j\in\{1,\dots,n\}$ such that*

$$\text{(2.1)}\quad (\operatorname{Re}\Gamma-\operatorname{Re}\langle x,y_j\rangle)(\operatorname{Re}\langle x,y_j\rangle-\operatorname{Re}\gamma) + (\operatorname{Im}\Gamma-\operatorname{Im}\langle x,y_j\rangle)(\operatorname{Im}\langle x,y_j\rangle-\operatorname{Im}\gamma)\geq 0$$

or, equivalently,

$$\text{(2.2)}\quad \left|\langle x,y_j\rangle-\frac{\gamma+\Gamma}{2}\right|\leq\frac{1}{2}|\Gamma-\gamma|$$

for each $j\in\{1,\dots,n\}$. Then

$$\text{(2.3)}\quad \left(\sum_{j=1}^{n}|\langle x,y_j\rangle|^2\right)^{\frac{1}{2}}\leq\frac{1}{\sqrt{n}}\|x\|\left\|\sum_{j=1}^{n}y_j\right\|+\frac{1}{4}\sqrt{n}\cdot\frac{|\Gamma-\gamma|^2}{|\Gamma+\gamma|}.$$

The equality holds in (2.3) for $x\neq0$ if and only if the equality case holds in (2.2) for each $j\in\{1,\dots,n\}$ and

$$\text{(2.4)}\quad \frac{1}{n}\sum_{j=1}^{n}y_j=\frac{1}{4}\cdot\frac{|\Gamma|^2+6\operatorname{Re}(\Gamma\bar{\gamma})+|\gamma|^2}{(\Gamma+\gamma)\|x\|^2}\cdot x.$$

Proof. The equivalence between (2.1) and (2.2) is obvious since for the complex numbers z,γ,Γ the following statements are equivalent:

(i) $\operatorname{Re}[(\Gamma-z)(\bar{z}-\bar{\gamma})]\geq0$;
(ii) $\left|z-\frac{\gamma+\Gamma}{2}\right|\leq\frac{1}{2}|\Gamma-\gamma|$.

The inequality (2.2) is clearly equivalent to

$$\text{(2.5)}\quad |\langle x,y_j\rangle|^2+\left|\frac{\gamma+\Gamma}{2}\right|^2\leq\frac{1}{4}|\Gamma-\gamma|^2+\operatorname{Re}\left[\left(\bar{\Gamma}+\bar{\gamma}\right)\langle x,y_j\rangle\right]$$

for $j\in\{1,\dots,n\}$ with equality iff the case of equality is realised in (2.2).

Summing over j from 1 to n in (2.5), we get:

$$\text{(2.6)}\quad \sum_{j=1}^{n}|\langle x,y_j\rangle|^2+n\left|\frac{\gamma+\Gamma}{2}\right|^2\leq\frac{1}{4}n|\Gamma-\gamma|^2+\operatorname{Re}\left[\left(\bar{\Gamma}+\bar{\gamma}\right)\left\langle x,\sum_{j=1}^{n}y_j\right\rangle\right]$$

with equality if and only if the equality case holds for each j in (2.2).

Utilising the arithmetic mean – geometric mean inequality we can state

$$\text{(2.7)}\quad 2\sqrt{n}\left|\frac{\gamma+\Gamma}{2}\right|\left(\sum_{j=1}^{n}|\langle x,y_j\rangle|^2\right)^{\frac{1}{2}}\leq\sum_{j=1}^{n}|\langle x,y_j\rangle|^2+n\left|\frac{\gamma+\Gamma}{2}\right|^2$$

with equality if and only if

$$\text{(2.8)}\quad \sum_{j=1}^{n}|\langle x,y_j\rangle|^2=n\cdot\left|\frac{\gamma+\Gamma}{2}\right|^2.$$

Combining (2.6) with (2.7) we deduce

$$\left(\sum_{j=1}^{n}|\langle x,y_j\rangle|^2\right)^{\frac{1}{2}} \le \frac{1}{4}\sqrt{n}\cdot\frac{|\Gamma-\gamma|^2}{|\Gamma+\gamma|}+\operatorname{Re}\left[\frac{(\bar{\Gamma}+\bar{\gamma})}{\Gamma+\gamma}\left\langle x,\sum_{j=1}^{n}y_j\right\rangle\right] \tag{2.9}$$
$$\le \frac{1}{4}\sqrt{n}\cdot\frac{|\Gamma-\gamma|^2}{|\Gamma+\gamma|}+\left|\left\langle x,\sum_{j=1}^{n}y_j\right\rangle\right|$$
$$\le \frac{1}{4}\sqrt{n}\cdot\frac{|\Gamma-\gamma|^2}{|\Gamma+\gamma|}+\|x\|\left\|\sum_{j=1}^{n}y_j\right\|.$$

For the last inequality in (2.9) we have used Schwarz's inequality $|\langle u,v\rangle|\le\|u\|\,\|v\|$, $u,v\in H$, for which, since

$$\left\|u-\frac{\langle u,v\rangle v}{\|v\|^2}\right\|^2=\frac{\|u\|^2\|v\|^2-|\langle u,v\rangle|^2}{\|v\|^2},\quad v\neq 0,$$

the equality case holds if and only if

$$u=\frac{\langle u,v\rangle v}{\|v\|^2}.$$

Therefore the equality case holds in the last part of (2.9) if and only if

$$\sum_{j=1}^{n}y_j=\frac{\sum_{j=1}^{n}\overline{\langle x,y_j\rangle}x}{\|x\|^2}.$$

Now, if the equality case holds in (2.2) for each $j\in\{1,\dots,n\}$, then squaring and summing over $j\in\{1,\dots,n\}$, we deduce:

$$\sum_{j=1}^{n}|\langle x,y_j\rangle|^2=\operatorname{Re}\left\langle x,(\gamma+\Gamma)\sum_{j=1}^{n}y_j\right\rangle+\frac{1}{4}n|\Gamma-\gamma|^2-\frac{1}{4}n|\Gamma+\gamma|^2. \tag{2.10}$$

From (2.4), taking the inner product and the real part we have

$$\operatorname{Re}\left\langle x,(\gamma+\Gamma)\sum_{j=1}^{n}y_j\right\rangle=\frac{n}{4}\cdot\frac{|\Gamma|^2+6\operatorname{Re}(\Gamma\bar{\gamma})+|\gamma|^2}{\|x\|^2}\|x\|^2 \tag{2.11}$$
$$=\frac{n}{4}\left[2|\Gamma+\gamma|^2-|\Gamma-\gamma|^2\right].$$

Therefore, by (2.10) and (2.11) we get

$$\sum_{j=1}^{n}|\langle x,y_j\rangle|^2=\frac{1}{4}n\left[2|\Gamma+\gamma|^2-|\Gamma-\gamma|^2\right]+\frac{1}{4}n|\Gamma-\gamma|^2-\frac{1}{4}n|\Gamma+\gamma|^2 \tag{2.12}$$
$$=\frac{n}{4}|\Gamma+\gamma|^2.$$

Taking the norm in (2.4) we have

$$\begin{aligned}(2.13)\qquad & \frac{1}{\sqrt{n}}\|x\|\left\|\sum_{j=1}^{n}y_j\right\|+\frac{1}{4}\sqrt{n}\cdot\frac{|\Gamma-\gamma|^2}{|\Gamma+\gamma|}\\ &=\frac{\sqrt{n}}{4}\cdot\frac{\left(2|\Gamma+\gamma|^2-|\Gamma-\gamma|\right)}{|\Gamma+\gamma|}+\frac{1}{4}\sqrt{n}\cdot\frac{|\Gamma-\gamma|^2}{|\Gamma+\gamma|}\\ &=\frac{\sqrt{n}}{2}|\Gamma+\gamma|.\end{aligned}$$

The equations (2.12) and (2.13) show that the equality case is realised in (2.3). Conversely, if the equality case holds in (2.3), it must hold in all inequalities needed to prove it, therefore, we must have:

$$(2.14)\qquad \left|\langle x,y_j\rangle-\frac{\Gamma+\gamma}{2}\right|=\frac{1}{2}|\Gamma-\gamma| \quad \text{for each} \quad j\in\{1,\dots,n\},$$

$$(2.15)\qquad \sum_{j=1}^{n}|\langle x,y_j\rangle|^2=n\left|\frac{\gamma+\Gamma}{2}\right|^2,$$

$$(2.16)\qquad \operatorname{Im}\left\langle x,(\Gamma+\gamma)\sum_{j=1}^{n}y_j\right\rangle=0$$

and

$$(2.17)\qquad \sum_{j=1}^{n}y_j=\frac{\overline{\left\langle x,\sum_{j=1}^{n}y_j\right\rangle}}{\|x\|^2}\cdot x.$$

From (2.14) we get:

$$\operatorname{Re}\langle x,(\Gamma+\gamma)y_j\rangle=|\langle x,y_j\rangle|^2+\frac{1}{4}|\Gamma+\gamma|^2-\frac{1}{4}|\Gamma-\gamma|^2,$$

which, by summation over j and (2.15), gives

$$\begin{aligned}(2.18)\qquad \operatorname{Re}\left\langle x,(\Gamma+\gamma)\sum_{j=1}^{n}y_j\right\rangle &=n\left[\left|\frac{\Gamma+\gamma}{2}\right|^2-\left|\frac{\Gamma-\gamma}{2}\right|^2\right]\\ &=\frac{n}{4}\cdot\left[|\Gamma|^2+6\operatorname{Re}(\Gamma\bar{\gamma})+|\gamma|^2\right].\end{aligned}$$

On multiplying (2.17) by $\gamma+\Gamma\neq 0$, we have

$$\begin{aligned}(2.19)\quad (\Gamma+\gamma)\cdot\sum_{j=1}^{n}y_j &=\frac{\overline{\left\langle x,(\Gamma+\gamma)\sum_{j=1}^{n}y_j\right\rangle}}{\|x\|^2}\cdot x\\ &=\frac{\operatorname{Re}\left\langle x,(\Gamma+\gamma)\sum_{j=1}^{n}y_j\right\rangle-i\operatorname{Im}\left\langle x,(\Gamma+\gamma)\sum_{j=1}^{n}y_j\right\rangle}{\|x\|^2}\cdot x.\end{aligned}$$

Finally, on making use of (2.16), (2.18) and (2.19), we deduce the equality (2.4) and the proof is complete. ∎

The following results that provide a different bound for the Bessel sum $\sum_{j=1}^{n}|\langle x,y_j\rangle|^2$ may be stated as well.

Theorem 10. *Let $(H;\langle\cdot,\cdot\rangle)$ be a real or complex inner product space, $\Gamma,\gamma\in\mathbb{K}$ with $\operatorname{Re}(\Gamma\bar{\gamma})>0$ and $x,y_j\in H$, $j\in\{1,\dots,n\}$ such that either (2.1) or (2.2) hold true. Then*

$$\sum_{j=1}^{n}|\langle x,y_j\rangle|^2\le\frac{1}{n}\cdot\frac{|\Gamma+\gamma|^2}{4\operatorname{Re}(\Gamma\bar{\gamma})}\left\|\sum_{j=1}^{n}y_j\right\|^2\|x\|^2. \tag{2.20}$$

The equality holds in (2.20) for $x\neq0$ if and only if the equality case holds in (2.2) for each $j\in\{1,\dots,n\}$ and

$$\frac{1}{n}\cdot\sum_{j=1}^{n}y_j=\frac{2\operatorname{Re}(\Gamma\bar{\gamma})}{(\Gamma+\gamma)\|x\|^2}\cdot x. \tag{2.21}$$

Proof. From (2.6) we have

$$\sum_{j=1}^{n}|\langle x,y_j\rangle|^2+n\operatorname{Re}(\Gamma\bar{\gamma})\le\operatorname{Re}\left[\left(\bar{\Gamma}+\bar{\gamma}\right)\left\langle x,\sum_{j=1}^{n}y_j\right\rangle\right] \tag{2.22}$$

with equality if and only if the case of equality holds in (2.2) for each $j\in\{1,\dots,n\}$.

Utilising the elementary inequality between the arithmetic and geometric mean, we have

$$2\sqrt{n}\left(\sum_{j=1}^{n}|\langle x,y_j\rangle|^2\right)^{\frac{1}{2}}\sqrt{\operatorname{Re}(\Gamma\bar{\gamma})}\le\sum_{i=1}^{n}|\langle x,e_i\rangle|^2+n\operatorname{Re}(\Gamma\bar{\gamma}) \tag{2.23}$$

with equality if and only if

$$\sum_{j=1}^{n}|\langle x,y_j\rangle|^2=n\operatorname{Re}(\Gamma\bar{\gamma}). \tag{2.24}$$

Combining (2.22) with (2.23) we deduce

$$\begin{aligned}\sum_{j=1}^{n}|\langle x,y_j\rangle|^2&\le\frac{\left\{\operatorname{Re}\left[\left(\bar{\Gamma}+\bar{\gamma}\right)\left\langle x,\sum_{j=1}^{n}y_j\right\rangle\right]\right\}^2}{4n\operatorname{Re}(\Gamma\bar{\gamma})}\\&\le\frac{|\Gamma+\gamma|^2\left|\left\langle x,\sum_{j=1}^{n}y_j\right\rangle\right|^2}{4n\operatorname{Re}(\Gamma\bar{\gamma})}\\&\le\frac{1}{n}\cdot\frac{|\Gamma+\gamma|^2}{4\operatorname{Re}(\Gamma\bar{\gamma})}\left\|\sum_{j=1}^{n}y_j\right\|^2\|x\|^2,\end{aligned} \tag{2.25}$$

where, for the last inequality we have used Schwarz's inequality. The equality case holds in the last inequality (2.25) iff

$$\sum_{j=1}^{n}y_j=\frac{\overline{\left\langle x,\sum_{j=1}^{n}y_j\right\rangle}\,x}{\|x\|^2}. \tag{2.26}$$

Now, if the equality case holds in (2.2) for each $j \in \{1, \dots, n\}$, then squaring and summing over $j \in \{1, \dots, n\}$ we deduce:

$$\sum_{j=1}^{n} |\langle x, y_j \rangle|^2 = \operatorname{Re} \left\langle x, (\Gamma + \gamma) \sum_{j=1}^{n} y_j \right\rangle + \frac{1}{4} n |\Gamma - \gamma|^2 - \frac{1}{4} n |\Gamma + \gamma|^2 . \tag{2.27}$$

From (2.21), taking the inner product and the real part we have

$$\operatorname{Re} \left\langle x, (\Gamma + \gamma) \sum_{j=1}^{n} y_j \right\rangle = 2n \operatorname{Re} (\Gamma \bar{\gamma}) . \tag{2.28}$$

Therefore, by (2.27) and (2.28) we have

$$\sum_{j=1}^{n} |\langle x, y_j \rangle|^2 = n \operatorname{Re} (\Gamma \bar{\gamma}) .$$

Taking the norm in (2.21) we have

$$\frac{1}{n} \cdot \frac{|\Gamma + \gamma|^2}{4 \operatorname{Re} (\Gamma \bar{\gamma})} \left\| \sum_{j=1}^{n} y_j \right\|^2 \|x\|^2 = n \operatorname{Re} (\Gamma \bar{\gamma}) .$$

showing that the equality case holds true in (2.20).

Conversely, if the equality case holds in (2.20), it must hold in all inequalities needed to prove it, therefore we must have (2.14), (2.24), (2.26) and

$$\operatorname{Im} \left\langle x, (\Gamma + \gamma) \sum_{j=1}^{n} y_j \right\rangle = 0 \tag{2.29}$$

From (2.14) we have

$$\operatorname{Re} \langle x, (\Gamma + \gamma) y_j \rangle = |\langle x, y_j \rangle|^2 + \frac{1}{4} |\Gamma + \gamma|^2 - \frac{1}{4} |\Gamma - \gamma|^2 ,$$

which by summation over j and (2.24) gives

$$\operatorname{Re} \left\langle x, (\Gamma + \gamma) \sum_{j=1}^{n} y_j \right\rangle = n \operatorname{Re} (\Gamma \bar{\gamma}) + n \operatorname{Re} (\Gamma \bar{\gamma}) = 2n \operatorname{Re} (\Gamma \bar{\gamma}) . \tag{2.30}$$

Since $\gamma + \Gamma$ is not zero (because $|\Gamma + \gamma|^2 \geq 4 \operatorname{Re} (\Gamma \bar{\gamma}) > 0$), we have by (2.26), (2.29) and (2.30) that

$$\begin{aligned} \sum_{j=1}^{n} y_j &= \frac{\operatorname{Re} \left\langle x, (\Gamma + \gamma) \sum_{j=1}^{n} y_j \right\rangle - i \operatorname{Im} \left\langle x, (\Gamma + \gamma) \sum_{j=1}^{n} y_j \right\rangle}{(\Gamma + \gamma) \|x\|^2} \cdot x \\ &= \frac{2n \operatorname{Re} (\Gamma \bar{\gamma})}{(\Gamma + \gamma) \|x\|^2} \cdot x, \end{aligned} \tag{2.31}$$

which obviously imply the identity (2.21), and the proof of the theorem is complete. ∎

Remark 1. *A more convenient sufficient condition for (2.1) to hold is*

$$\operatorname{Re} \Gamma \geq \operatorname{Re} \langle x, y_j \rangle \geq \operatorname{Re} \gamma \quad \text{and} \quad \operatorname{Im} \Gamma \geq \operatorname{Im} \langle x, y_j \rangle \geq \operatorname{Im} \gamma \tag{2.32}$$

for each $j \in \{1, \dots, n\}$.

Remark 2. *If $\{e_1, \dots, e_n\}$ is an orthonormal family of vectors, then from (2.3) we get*

$$\left(\sum_{j=1}^{n} |\langle x, e_j \rangle|^2 \right)^{\frac{1}{2}} \leq \|x\| + \frac{1}{4} \sqrt{n} \frac{|\Gamma - \gamma|^2}{|\Gamma + \gamma|} \tag{2.33}$$

while from (2.20) we get

$$\sum_{j=1}^{n} |\langle x, e_j \rangle|^2 \leq \frac{|\Gamma + \gamma|}{4 \operatorname{Re}(\Gamma \overline{\gamma})} \|x\|^2 . \tag{2.34}$$

One must observe that in this case both (2.33) and (2.34) provide coarser bounds than Bessel's inequality (1.1). Therefore, since (2.3) and (2.20) are sharp inequalities, they must be used only in the case of non-orthogonal vectors satisfying (2.1) or (2.2).

3. Applications for Complex Numbers

Utilising Theorems 9 and 10 above, one can sate the following reverses of the generalised triangle inequality for complex numbers that may be of interest in applications:

If

$$(\operatorname{Re} \Gamma - \operatorname{Re} z_j) (\operatorname{Re} z_j - \operatorname{Re} \gamma) + (\operatorname{Im} \Gamma - \operatorname{Im} z_j) (\operatorname{Im} z_j - \operatorname{Im} \gamma) \geq 0 \tag{3.1}$$

or, equivalently,

$$\left| z_j - \frac{\gamma + \Gamma}{2} \right| \leq \frac{1}{2} |\Gamma - \gamma| \tag{3.2}$$

for each $j \in \{1, \dots, n\}$, then

$$\left(\sum_{j=1}^{n} |z_j|^2 \right)^{1/2} \leq \frac{1}{\sqrt{n}} \left| \sum_{j=1}^{n} z_j \right| + \frac{1}{4} \sqrt{n} \frac{|\Gamma - \gamma|^2}{|\Gamma + \gamma|} \tag{3.3}$$

provided $\Gamma \neq -\gamma$, and

$$\sum_{j=1}^{n} |z_j|^2 \leq \frac{1}{n} \cdot \frac{|\Gamma + \gamma|^2}{4 \operatorname{Re}(\Gamma \overline{\gamma})} \left| \sum_{j=1}^{n} z_j \right|^2 \tag{3.4}$$

provided $\operatorname{Re}(\Gamma \overline{\gamma}) > 0$.

The equality holds in (3.3) if and only if the equality case holds in (3.1) (or in (3.2)) for each $j \in \{1, \dots, n\}$ and

$$\frac{1}{n} \sum_{j=1}^{n} z_j = \frac{1}{4} \cdot \frac{|\Gamma|^2 + 6 \operatorname{Re}(\Gamma \overline{\gamma}) + |\gamma|^2}{\overline{\Gamma} + \overline{\gamma}}.$$

The equality holds in (3.4) if and only if the equality case holds in (3.1) (or in (3.2)) for each $j \in \{1, \dots, n\}$ and

$$\frac{1}{n} \sum_{j=1}^{n} z_j = \frac{2 \operatorname{Re}(\Gamma \overline{\gamma})}{\overline{\Gamma} + \overline{\gamma}}.$$

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