On Huygen‘s Trigonometric Inequality

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1. Introduction

Let $x \in \left(0, \frac{\pi}{2}\right)$. Then it is well known that $\sin x < x < tgx$, used for the first time by Archimede in the numerical approximation of π. Nicolaus de Cusa (1401-1464), by using certain geometrical constructions, discovered the relation:

$$\frac{3\sin x}{2+\cos x} < x \tag{1}$$

In 1631, Willebrod Snellius (1581-1626), in his book entitled "Cyclometicus" found a proof for (1), and for the following inequality:

$$2\sin x + tgx > 3x \tag{2}$$

The proofs given by Snellius were quite obscure, but fortunately these formulae are true. The first scientist who found an acceptable (geometrical) proof for (1) and (2), after 33 years from the publication of Snellius‘ book, was Christian Huygens (1629-1695). Huygens, in his book "De circuli magnitudine inventa" usend (1) and (2) in the approximation of values of π. For the history of such themes, connected also to the fabulous history of the number π, see the References (especially [1-4]).

In what follows, we will call (1) (like e.g. in [1]) as Cusa‘s inequality, while (2) as the Huygen‘s inequality. In paper [5] the first author proved the following generalization of Cusa‘s inequality: Let $a, b, c > 0$ such that $2b \le c \le a + b$. Then, for any $x \in \left(0, \frac{\pi}{2}\right)$ one has

$$\frac{c\sin x}{a+b\cos x} < x \tag{3}$$

Particularly, for $c = 3$, $a = 2$, $b = 1$ we reobtain (1).

In problem 2585 of Crux Mathematicorum, V.N. Murty proposed the inequality:

$$tgx + \sin x > 2x \tag{4}$$

We note that Huygen‘s inequality (2) is stronger that (4) , as

$$\frac{tgx + \sin x}{2} > \frac{2\sin x + tgx}{3} > x \tag{5}$$

where the first relation is equivalent to $tgx > \sin x$.

The aim of this note is to offer certain generalizations to Huygens-like inequalities.

2. Main Result

First we state the following:

Theorem 1. Let $a, b, c > 0$. The inequality

$$a \sin x + b\ tgx \geq c \cdot tg\frac{x}{2}, x \in \left(0, \frac{\pi}{2}\right) \tag{6}$$

holds true only in the following cases:

$$\begin{array}{l} \text{i). } \triangle = (b-a)^2 - 4k(a+b-k) \leq 0, \text{ where } k = \frac{c}{2}; \\ \text{ii). } \triangle > 0, \text{ and } a - b \geq 2k = c; \\ \text{iii). } \triangle > 0, \text{ and } a - b \leq 0 \end{array} \tag{7}$$

Proof. Put $tg\frac{x}{2} = t$, and remark that for $x \in \left(0, \frac{\pi}{2}\right)$ one has $t \in (0,1)$. By $\sin x = 2t/\left(1+t^2\right), tgx = 2t/\left(1-t^2\right)$, after elementary transformation, (6) becomes

$$f(u) = ku^2 + u(b-a) + a + b - k \geq 0 \text{ for all } u \in (0,1), \tag{0.1}$$

where $k = \frac{c}{2}$ and $u = t^2 \in (0,1)$.

The discriminant of $f(a) = 0$ is $\triangle = (b-a)^2 - 4k(a+b-k)$. Since $k > 0$, if $\triangle \leq 0$, then it is well known that $f(a) \geq 0$ for all real u, particularly for $u \in (0,1)$. Thus i). follows:

Now, if $\triangle > 0$, the roots of $f(u) = 0$ are $u_1 = \frac{a-b-\sqrt{\triangle}}{2k}$, $u_2 = \frac{a-b+\sqrt{\triangle}}{2k}$. For $a - b > 0$ we have $0 < u_1 < u_2$. Now, $f(1) = 2b > 0$, and we must have $f(0) = a + b - k \geq 0$. Clearly, we cannot have $u_1 \leq 1$, since f being continuous, f would change signs on $(0,1)$. The inequalityu $u_1 > 1$ is equivalent to $a - b - 2k > \sqrt{\triangle}$. This is impossible if $a - b - 2k < 0$, but for $a - b - 2k \geq 0$, it becomes $(a-b-2k)^2 \geq \triangle$, so $(a-b)^2 - 4k(a-b) + 4k^2 \geq (a-b)^2 - 4k(a+b-k)$, i.e. $8kb > 0$, which is true. Therefore, ii). follows.

In case $a - b \leq 0$, one has $u_1 < u_2 < 0$, so $f(u) > 0$ for any $u \in (0,1)$.

Corollary. If one of $i) - iii)$ of (7) is satisfied, then

$$a \sin x + b\ tgx > \frac{c}{2}x, \text{ for all } x \in \left(0, \frac{\pi}{2}\right) \tag{9}$$

Proof. By $tg\frac{x}{2} > \frac{x}{2}$, (6) implies relation (9).

Examples. 1). Let $a = b = 1, c = 4$, in (6). Since i). is true, we get: (10).

$$\sin x + tgx \geq 4tg\frac{x}{2} > 2x \tag{10}$$

This improves also (4).

2). Let $a = \frac{1}{4}, b = \frac{3}{4}, c = 2$. Then iii). applies, and one gets:

$$\sin x + 3tgx \geq 8tg\frac{x}{2} > 4x \tag{11}$$

Theorem 2. Let $a, b > 0$, and put $A = \frac{b+\sqrt{b^2+4ab}}{2a}$.
If $\cos x \leq A$, $x \in \left[0, \frac{\pi}{2}\right)$, then

$$a \sin x + b\ tgx \geq (a+b)x \tag{12}$$

If $A \leq 1$ and $\cos x \geq A$, for $x \in \left[0, \frac{\pi}{2}\right)$, then

$$a \sin x + b \ tgx \leq (a+b)\, x \tag{13}$$

Proof. Let $f(x) = a \sin x + b \ tgx - (a+b)\, x,\ x \in \left[0, \frac{\pi}{2}\right)$. Since the derivative of f is $f'(x) = \frac{1}{\cos^2 x}\left[a \cos^3 x - (a+b)\cos^2 x + b\right] = g(t)/t^2$, where $t = \cos x \in (0,1]$ and $g(t) = at^3 - (a+b)\, t^2 + b$, we have to study the signs of $f'(x)$. By $g(t) = at^3 - at^2 - bt^2 + b = at^2(t-1) - b\left(t^2-1\right) = (t-1)\left(at^2 - bt - b\right)$ and remarking that $at^2 - bt - b = 0$ has as roots $t_1 = \frac{b-\sqrt{b^2+4ab}}{2a} < 0$ and $t_2 = A$, we can write $g(t) = a(t-t_1)(t-A) \leq 0$ if $t = \cos x \leq A$. By $t - 1 \leq 0$, it follows $g(t) \geq 0$, so f is an increasing function on $\left[0, \frac{\pi}{2}\right)$. Clearly, if $\cos x < A$ and $t < 1$ (i.e. $x \in \left(0, \frac{\pi}{2}\right)$), then f is strictly increasing. This implies $f(x) \geq f(0) = 0$, so relation (12) follows. A similar proof applies to (13), and we omit the details.

Examples. 1). Let $a = 2, b = 1$. Then $A = 1$, so $\cos x < A$ in $\left(0, \frac{\pi}{2}\right)$. Then (12) reduces to Huygen's inequality (2).

2). Put $a = 6, b = 1$. Then $a = \frac{1}{2}$. The inequality $\cos x > \frac{1}{2}$ is true for $x \in \left(0, \frac{\pi}{3}\right)$. By relation (13) we get

$$6 \sin x + tgx < 7x \text{ for } x \in \left(0, \frac{\pi}{3}\right) \tag{14}$$

Thus (2) and (14) can be written in a single line as

$$\frac{6 \sin x + tgx}{7} < x < \frac{2 \sin x + tgx}{3}, x \in \left(0, \frac{\pi}{3}\right) \tag{15}$$

Inequality (14) is reversed for $x \in \left(\frac{\pi}{3}, \frac{\pi}{2}\right)$.

Remarks.

(1) More generally, an inequality of type

$$a \sin x + b \ tgx \geq cx \tag{16}$$

can be studied. In this case, by letting $f(x) = a \sin x + b \ tgx - cx$, one obtains $f'(x) = g(t)/t^2$, where $g(t) = at^3 - ct^2 + b$. Since $g'(t) = t(3at - 2c)$, the study of sign changes of g depens on $g\left(\frac{2c}{3a}\right) = \left(27a^2 b - 4c^3\right)/27a^2$. But then there will be more cases, with much more complicated (and not so nice) results than Theorem 2.

(2) Other geometric inequalities (like Jordan's, Redheffer's, Kober's, etc.) can be found in [6] and [7]. See also [8] (where there are not included Cusa or Huygens type inequalities).

(3) For trigonometric inequalities, based on geometrical constructions, see [6] (one of them included also in [9]), and on convexity methods, see [7]. For other geometric inequalities see [10].

(4) If $a, b > 0$ and $x \in \left(0, \frac{\pi}{2}\right)$, then $a^2 tgx + b^2 \sin x > 2abx$.

Proof. We take $f(x) = a^2tgx + b^2\sin x - 2abx$, because $f'(x) > \left(\frac{a}{\cos x} - b\cos x\right)^2 \geq 0$, and $f(0) = 0$, so f is increasing and because is continuous, therefore $f(x) > f(0) = 0$, for all $x \in \left(0, \frac{\pi}{2}\right)$.

(5) Using the method from 4). for all $a, b > 0$ and $x \in \left(0, \frac{\pi}{2}\right)$, holds

$$a^2tgx + b^2\left(\frac{x}{2} + \frac{\sin 2x}{4}\right) > 2abx.$$

(6) We propose to study the following inequality

$$a^2tgx + \frac{b^2x}{2} > 2ab\ln\left(\frac{1}{\cos x}\right) + \frac{b^2\sin 2x}{4},$$

for all $a, b > 0$ and $x \in \left(0, \frac{\pi}{2}\right)$.

(7) If $x \in \left(0, \frac{\pi}{2}\right)$, then $\frac{\sin x}{x} > \sqrt{\cos x}$.

Proof. If $f(x) = \ln(tgx) + \ln(\sin x) - 2\ln x$, then

$$f'(x) = \frac{x(1+cos^2x) - 2\sin x\cos x}{\sin x\cos x} > \frac{2x\cos x - 2\sin x\cos x}{\sin x\cos x}$$
$$= \frac{2\cos x(x - \sin x)}{\sin x\cos x} > 0$$

etc.

(8) We propose to prove that for all $x \in \left(0, \frac{\pi}{2}\right)$ and all $\alpha > 0$ holds $\left(\frac{\sin x}{x}\right)^\alpha > \frac{\cos^\alpha x}{1+\cos^\alpha x}$.

(9) We propose to study the following inequality $a(\sin x)^\alpha + b(tgx)^\beta > cx^\gamma$, where $a, b, c, \alpha, \beta, \gamma \in R$ and $x \in \left(0, \frac{\pi}{2}\right)$.

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