SOME INEQUALITIES FOR A SINGULAR INTEGRAL

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Abstract. Estimates for a singular integral for which the Grüss, Čebyšev, Lupaş and Ostrowski inequalities fail to work are given by utilising trapezoidal type rules.

1. Introduction

Define the integral

$$I_p(f) := \int_a^b \frac{f(t)}{(t-a)^p}\,dt, \quad p \in (0,1);$$

when $f : [a,b] \to \mathbb{R}$ is Lebesgue measurable on $[a,b]$ and the Lebesgue integral is finite for any given $p \in (0,1)$.

In order to obtain estimates for this integral one may use the following refinement of the Grüss inequality obtained by Cheng and Sun in [1]:

$$\left| \frac{1}{b-a}\int_a^b f(t)\,g(t)\,dt - \frac{1}{b-a}\int_a^b f(t)\,dt \cdot \frac{1}{b-a}\int_a^b g(t)\,dt \right| \leq \frac{1}{2}(M-m)\frac{1}{b-a}\int_a^b \left| g(t) - \frac{1}{b-a}\int_a^b g(s)\,ds \right| dt,$$

provided that $f, g : [a,b] \to \mathbb{R}$ are Lebesgue measurable and

$$-\infty < m \leq f(t) \leq M < \infty$$

for a.e. $t \in [a,b]$.

For an extension to the general case of Lebesgue integrals on measurable spaces and for the sharpness of the constant $\frac{1}{2}$, see [2].

For this type of integral, the classical Grüss inequality,

$$\left| \frac{1}{b-a}\int_a^b f(t)\,g(t)\,dt - \frac{1}{b-a}\int_a^b f(t)\,dt \cdot \frac{1}{b-a}\int_a^b g(t)\,dt \right| \leq \frac{1}{4}(M-m)(N-n)$$

where

$$-\infty < m \leq f(t) \leq M < \infty, -\infty < n \leq g(t) \leq N < \infty$$

for a.e. $t \in [a,b]$, cannot be used since the second function, namely $g(t) = 1/(t-a)^p$, $p \in (0,1)$ is not essentially bounded on $[a,b]$.

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The same can be said for attemps to use the Čebyšev [3, p. 297], Lupaş [3, p. 301] or Ostrowski [3, p. 300] inequalities that provide bounds for the quantity

$$\left|\frac{1}{b-a}\int_a^b f(t)\,g(t)\,dt-\frac{1}{b-a}\int_a^b f(t)\,dt\cdot\frac{1}{b-a}\int_a^b g(t)\,dt\right|$$

in terms of the derivatives of the functions f,g.

The details are left as an exercise to the reader.

The main aim of this paper is to point out various estimates for $I_p(f)$ using an approach which employs the removal of the singularity and the utilisation of the trapezoidal rule.

2. Some Inequalities

The following result that establishes an estimate of the singular integral $I_p(f)$ in terms of the function values at the points a and b and in terms of the derivative at the point a holds true:

Theorem 1. *Let $f:[a,b]\to\mathbb{R}$ be a differentiable function so that f' is absolutely continuous on $[a,b]$, then we have the inequalities:*

$$\left|I_p(f)-\frac{2f(a)+(1-p)f(b)}{(3-p)(1-p)}(b-a)^{1-p}-\frac{1}{(3-p)(2-p)}f'(a)(b-a)^{2-p}\right| \tag{2.1}$$

$$\leq\begin{cases}\dfrac{1}{2(3-p)}\int_a^b(t-a)^{2-p}\|f''\|_{[a,t],\infty}\,dt & \text{if } f''\in L_\infty[a,b];\\[2ex] \dfrac{1}{(3-p)(\beta+1)^{\frac{1}{\beta}}}\int_a^b(t-a)^{1+\frac{1}{\beta}-p}\|f''\|_{[a,t],\alpha}\,dt & \text{if } f''\in L_\alpha[a,b],\\ & \alpha>1,\ \frac{1}{\alpha}+\frac{1}{\beta}=1;\\[2ex] \dfrac{1}{(3-p)}\int_a^b(t-a)^{1-p}\|f''\|_{[a,t],1}\,dt; & \end{cases}$$

$$\leq\begin{cases}\dfrac{1}{2(3-p)^2}(b-a)^{3-p}\|f''\|_{[a,b],\infty} & \text{if } f''\in L_\infty[a,b];\\[2ex] \dfrac{(b-a)^{2+\frac{1}{\beta}-p}}{(3-p)(\beta+1)^{\frac{1}{\beta}}\left(2+\frac{1}{\beta}-p\right)}\|f''\|_{[a,b],\alpha} & \text{if } f''\in L_\alpha[a,b],\\ & \alpha>1,\ \frac{1}{\alpha}+\frac{1}{\beta}=1;\\[2ex] \dfrac{(b-a)^{2-p}}{(2-p)(3-p)}\|f''\|_{[a,b],1}, & \end{cases}$$

for all $p\in(0,1)$.

Proof. We have

$$I_p(f)=\int_a^b\frac{f(t)}{(t-a)^p}dt=\int_a^b\frac{f(a)}{(t-a)^p}dt+\int_a^b\frac{f(t)-f(a)}{(t-a)^p}dt \tag{2.2}$$

$$=f(a)\frac{(b-a)^{1-p}}{1-p}+\int_a^b\frac{f(t)-f(a)}{(t-a)^p}dt.$$

Since f is absolutely continuous, then we have

$$f(t)-f(a)=\int_a^t f'(u)\,du.$$

Using the following elementary identity which can be obtained from the integration by parts formula

$$\int_\alpha^\beta g(u)\,du=\frac{g(\alpha)+g(\beta)}{2}(\beta-\alpha)+\int_\alpha^\beta\left(\frac{\alpha+\beta}{2}-u\right)g'(u)\,du, \tag{2.3}$$

provided that g is absolutely continuous on the interval $[\alpha,\beta]$, we may state that:

$$\int_a^t f'(u)\,du=\frac{f'(t)+f'(a)}{2}(t-a)+\int_a^t\left(\frac{a+t}{2}-u\right)f''(u)\,du. \tag{2.4}$$

Consequently,

$$\begin{aligned}\int_a^b\frac{f(t)-f(a)}{(t-a)^p}dt&=\int_a^b(t-a)^{1-p}\left[\frac{f'(t)+f'(a)}{2}\right]dt\\&\quad+\int_a^b\left[\frac{1}{(t-a)^p}\int_a^t\left(\frac{a+t}{2}-u\right)f''(u)\,du\right]dt.\end{aligned} \tag{2.5}$$

However,

$$\begin{aligned}&\int_a^b(t-a)^{1-p}\left[\frac{f'(t)+f'(a)}{2}\right]dt\\&=\frac{1}{2}\left[\int_a^b f'(a)(t-a)^{1-p}\,dt+\int_a^b(t-a)^{1-p}f'(t)\,dt\right]\\&=\frac{1}{2}\left[\frac{f'(a)(b-a)^{2-p}}{2-p}+f(t)(t-a)^{1-p}\Big|_a^b-(1-p)\int_a^b\frac{f(t)}{(t-a)^p}dt\right]\\&=\frac{1}{2}\left[\frac{f'(a)(b-a)^{2-p}}{2-p}+f(b)(b-a)^{1-p}-(1-p)I_p(f)\right]\\&=\frac{1}{2}f(b)(b-a)^{1-p}+\frac{1}{2}\cdot\frac{f'(a)(b-a)^{2-p}}{2-p}-\frac{1-p}{2}I_p(f).\end{aligned}$$

Using (2.2), we may write that:

$$\begin{aligned}I_p(f)&=f(a)\frac{(b-a)^{1-p}}{1-p}+\frac{1}{2}f(b)(b-a)^{1-p}\\&\quad+\frac{1}{2}\frac{f'(a)(b-a)^{2-p}}{2-p}-\frac{1-p}{2}I_p(f)+R[f],\end{aligned} \tag{2.6}$$

where

$$R[f]=\int_a^b\left[\frac{1}{(t-a)^p}\int_a^t\left(\frac{a+t}{2}-u\right)f''(u)\,du\right]dt.$$

From (2.6), we get:

$$I_p(f)\left(1+\frac{1-p}{2}\right)=f(a)\frac{(b-a)^{1-p}}{1-p}+\frac{1}{2}f(b)(b-a)^{1-p}+\frac{1}{2}\frac{f'(a)(b-a)^{2-p}}{2-p}+R[f],$$

which implies that

$$I_p(f)=\frac{1}{3-p}\left[2f(a)\frac{(b-a)^{1-p}}{1-p}+f(b)(b-a)^{1-p}+\frac{f'(a)(b-a)^{2-p}}{2-p}\right]+\frac{2}{3-p}R[f],$$

i.e.,

$$I_p(f)=\frac{2f(a)+(1-p)f(b)}{(3-p)(1-p)}(b-a)^{1-p}+\frac{f'(a)(b-a)^{2-p}}{(3-p)(2-p)}+\frac{2}{3-p}R[f]. \tag{2.7}$$

On the other hand, we have

$$|R[f]|\leq\int_a^b\left[\frac{1}{(t-a)^p}\left|\int_a^t\left(\frac{a+t}{2}-u\right)f''(u)\,du\right|\right]dt=:J.$$

However,

$$\left|\int_a^t\left(\frac{a+t}{2}-u\right)f''(u)\,du\right|\leq\|f''\|_{[a,t],\infty}\int_a^t\left|\frac{a+t}{2}-u\right|du=\frac{(t-a)^2}{4}\|f''\|_{[a,t],\infty},$$

by Hölder's integral inequality, we have:

$$\left|\int_a^t\left(\frac{a+t}{2}-u\right)f''(u)\,du\right|\leq\|f''\|_{[a,t],\alpha}\left(\int_a^t\left|\frac{a+t}{2}-u\right|^\beta du\right)^{\frac{1}{\beta}}=\frac{(t-a)^{1+\frac{1}{\beta}}}{2(\beta+1)^{\frac{1}{\beta}}}\|f''\|_{[a,t],\alpha},\quad\alpha>1,\ \frac{1}{\alpha}+\frac{1}{\beta}=1;$$

and finally,

$$\left|\int_a^t\left(\frac{a+t}{2}-u\right)f''(u)\,du\right|\leq\sup_{u\in[a,t]}\left|\frac{a+t}{2}-u\right|\int_a^t|f''(u)|\,du=\frac{(t-a)}{2}\|f''\|_{[a,t],1}$$

and then

$$J \leq \begin{cases} \dfrac{1}{4}\displaystyle\int_a^b (t-a)^{2-p} \|f''\|_{[a,t],\infty} \, dt, \\[2ex] \dfrac{1}{2(\beta+1)^{\frac{1}{\beta}}} \displaystyle\int_a^b (t-a)^{1+\frac{1}{\beta}-p} \|f''\|_{[a,t],\alpha} \, dt, \\[2ex] \dfrac{1}{2}\displaystyle\int_a^b (t-a)^{1-p} \|f''\|_{[a,t],1} \, dt, \end{cases}$$

and the first part of (2.1) is proved.

We now observe that,

$$\int_a^b (t-a)^{2-p} \|f''\|_{[a,t],\infty} \, dt \leq \|f''\|_{[a,b],\infty} \frac{(b-a)^{3-p}}{3-p},$$

$$\int_a^b (t-a)^{1+\frac{1}{\beta}-p} \|f''\|_{[a,t],\alpha} \, dt \leq \|f''\|_{[a,b],\alpha} \frac{(b-a)^{2+\frac{1}{\beta}-p}}{2+\frac{1}{\beta}-p}$$

and

$$\int_a^b (t-a)^{1-p} \|f''\|_{[a,t],1} \, dt \leq \|f''\|_{[a,b],1} \frac{(b-a)^{2-p}}{2-p},$$

which proves the last part of the theorem. ∎

Theorem 2. *Let $f : [a,b] \to \mathbb{R}$ be such that the second derivative f'' is absolutely continuous on $[a,b]$, then,*

$$\left| I_p(f) - \frac{2f(a) + (1-p) f(b)}{(3-p)(1-p)} (b-a)^{1-p} - \frac{1}{(3-p)(2-p)} f'(a) (b-a)^{2-p} \right| \tag{2.8}$$

$$\leq \begin{cases} \dfrac{1}{12}\displaystyle\int_a^b (t-a)^{3-p} \|f'''\|_{[a,t],\infty} \, dt & \text{if } f''' \in L_\infty[a,b]; \\[2ex] \dfrac{[B(\beta+1,\beta+1)]^{\frac{1}{\beta}}}{2} \displaystyle\int_a^b (t-a)^{2+\frac{1}{\beta}-p} \|f'''\|_{[a,t],\alpha} \, dt & \text{if } f''' \in L_\alpha[a,b], \\ & \alpha > 1, \ \frac{1}{\alpha} + \frac{1}{\beta} = 1; \\[2ex] \dfrac{1}{8}\displaystyle\int_a^b (t-a)^{2-p} \|f'''\|_{[a,t],1} \, dt; \end{cases}$$

$$\leq \begin{cases} \dfrac{1}{12(4-p)} (b-a)^{4-p} \|f'''\|_{[a,b],\infty} \\[2ex] \dfrac{[B(\beta+1,\beta+1)]^{\frac{1}{\beta}}}{2\left(3+\frac{1}{\beta}-p\right)} (b-a)^{3+\frac{1}{\beta}-p} \|f'''\|_{[a,b],\alpha}, \ \alpha > 1, \ \frac{1}{\alpha} + \frac{1}{\beta} = 1; \\[2ex] \dfrac{1}{8} (b-a)^{3-p} \|f'''\|_{[a,b],1}; \end{cases}$$

for all $p \in (0,1)$*, where* $B(\cdot,\cdot)$ *is Euler's Beta function,*

$$B(s+1,q+1) := \int_0^1 t^s (1-t)^q \, dt, \quad s,q > -1.$$

Proof. Using the elementary identity

$$\int_\alpha^\beta g(u)\,du = \frac{g(\alpha)+g(\beta)}{2}(\beta-\alpha) + \frac{1}{2}\int_\alpha^\beta (u-\alpha)(u-\beta)\, g''(u)\,du,$$

where $g : [\alpha,\beta] \to \mathbb{R}$ is such that the first derivative is absolutely continuous, we have,

$$\int_a^t f'(u)\,du = \frac{f'(t)+f'(a)}{2}(t-a) + \frac{1}{2}\int_a^t (u-a)(u-t)\, f'''(u)\,du.$$

Consequently,

$$\int_a^b \frac{f(t)-f(a)}{(t-a)^p}dt = \int_a^b (t-a)^{1-p}\left[\frac{f'(t)+f'(a)}{2}\right]dt + \frac{1}{2}\int_a^b \left[\frac{1}{(t-a)^p}\int_a^t (u-a)(u-t)\, f'''(u)\,du\right]dt.$$

By a similar argument to that in the proof of the above theorem, we have (see also (2.7)) that,

$$I_p(f) = \frac{2f(a)+(1-p)f(b)}{(3-p)(1-p)}(b-a)^{1-p} + \frac{1}{(3-p)(2-p)}f'(a)(b-a)^{2-p} + \frac{2}{3-p}R_1[f].$$

where

$$R_1[f] = \frac{1}{2}\int_a^b \left[\frac{1}{(t-a)^p}\int_a^t (u-a)(t-u)\, f'''(u)\,du\right]dt.$$

Now, we have

$$|R_1[f]| \le \frac{1}{2}\int_a^b \left[\frac{1}{(t-a)^p}\left|\int_a^t (u-a)(t-u)\, f'''(u)\,du\right|\right]dt =: K.$$

However,

$$\left|\int_a^t (u-a)(u-t)\, f'''(u)\,du\right| \le \|f'''\|_{[a,t],\infty}\int_a^t (u-a)(t-u)\,du = \|f'''\|_{[a,t],\infty}\frac{(t-a)^3}{6}.$$

Using Hölder's integral inequality, we have

$$\left|\int_a^t (u-a)(u-t)\, f'''(u)\,du\right| \le \|f'''\|_{[a,t],\alpha}\left(\int_a^t (u-a)^\beta (t-u)^\beta\,du\right)^{\frac{1}{\beta}} = \|f'''\|_{[a,t],\alpha}(t-a)^{2+\frac{1}{\beta}}\left[B(\beta+1,\beta+1)\right]^{\frac{1}{\beta}}$$

for $\alpha > 1$, $\frac{1}{\alpha}+\frac{1}{\beta} = 1$.

Also,

$$\left|\int_a^t (u-a)(u-t) f'''(u)\,du\right| \le \max_{u\in[a,t]}\{(t-u)(u-a)\}\int_a^t |f'''(u)|\,du$$
$$= \frac{(t-a)^2}{4}\|f'''\|_{[a,t],1}\,.$$

We can now state that

$$K \le \begin{cases} \dfrac{1}{12}\displaystyle\int_a^b (t-a)^{3-p}\|f'''\|_{[a,t],\infty}\,dt \\[2ex] \dfrac{1}{2}\left[B(\beta+1,\beta+1)\right]^{\frac{1}{\beta}}\displaystyle\int_a^b (t-a)^{2+\frac{1}{\beta}-p}\|f'''\|_{[a,t],\alpha}\,dt \\[2ex] \dfrac{1}{8}\displaystyle\int_a^b (t-a)^{2-p}\|f'''\|_{[a,t],1}\,dt \end{cases}$$

and the first part of (2.8) is proved.

Now, we observe that

$$\int_a^b (t-a)^{3-p}\|f'''\|_{[a,t],\infty}\,dt \le \|f'''\|_{[a,b],\infty}\cdot\frac{(b-a)^{4-p}}{4-p},$$
$$\int_a^b (t-a)^{2+\frac{1}{\beta}-p}\|f'''\|_{[a,t],\alpha}\,dt \le \|f'''\|_{[a,b],\alpha}\cdot\frac{(b-a)^{3+\frac{1}{\beta}-p}}{3+\frac{1}{\beta}-p}$$

and

$$\int_a^b (t-a)^{2-p}\|f'''\|_{[a,t],1}\,dt \le \|f'''\|_{[a,b],1}\,\frac{(b-a)^{3-p}}{3-p}$$

which proves the last part of the inequality (2.8). ∎

3. Applications

A natural application of the above results is to approximate the Euler Beta function:

$$B(p+1,q+1) := \int_0^1 t^p(1-t)^q\,dt,\quad p,q>-1 \tag{3.1}$$

in the case that $p\in(-1,0)$ and $q\in(0,\infty)$.

For this purpose, we consider the function $f:[0,1]\to\mathbb{R}$, $f(t)=(1-t)^q$ with $q\in(0,\infty)$.

We observe that

$$f'(t) = -q(1-t)^{q-1},\qquad f''(t) = q(q-1)(1-t)^{q-2}.$$

We also observe that

$$\|f''\|_{[0,1],\infty} = q(q-1)\quad\text{if}\quad q\ge 2,$$
$$\|f''\|_{[0,1],\alpha} = q(q-1)\left[\int_0^1 (1-t)^{(q-2)\alpha}\,dt\right]^{\frac{1}{\alpha}}$$
$$= \frac{q(q-1)}{\left[(q-2)\alpha+1\right]^{\frac{1}{\alpha}}}\quad\text{if}\quad (q-2)\alpha+1>0,\ \alpha>1,\ q>1$$

and

$$\|f''\|_{[0,1],1} = q(q-1)\int_0^1 (1-t)^{q-2}\,dt = q \quad \text{if} \quad q>1.$$

Applying Theorem 1 for f as above, we have the inequalities

$$\left|B(p+1,q+1) - \frac{2}{(3-p)(1-p)} + \frac{2}{(3-p)(2-p)}\right| \tag{3.2}$$

$$\leq \begin{cases} \dfrac{q(q-1)}{2(3-p)} \text{ if } p\in(-1,0),\ q\in(2,\infty); \\[2ex] \dfrac{1}{(3-p)(\beta+1)^{\frac{1}{\beta}}\left(2+\frac{1}{\beta}-p\right)}\cdot\dfrac{q(q-1)}{[(q-2)\alpha+1]^{\frac{1}{\alpha}}} \\ \quad \text{if } p\in(-1,0) \text{ and } q,\alpha>1, \text{with } (q-2)\alpha+1>0 \text{ and } \frac{1}{\alpha}+\frac{1}{\beta}=1; \\ \dfrac{q}{(2-p)(3-p)}, \text{ if } p\in(-1,0) \text{ and } q\in(1,\infty). \end{cases}$$

A similar result can be obtained by using Theorem 2. The details are omitted.

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