TWO CLASS OF COMPLETELY MONOTONIC FUNCTIONS INVOLVING GAMMA AND POLYGAMMA FUNCTIONS

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Abstract. The function

$$\frac{[\Gamma(x+1)]^{1/x}}{x^c}\left(1+\frac{1}{x}\right)^x$$

is logarithmically completely monotonic in $(0,\infty)$ if and only if $c \geq 1$ and its reciprocal is logarithmically completely monotonic in $(0,\infty)$ if and only if $c \leq 0$. The function

$$\psi''(x)+\frac{2+(6+c)x+(4+3c)x^2+(2+3c)x^3+cx^4}{x^3(x+1)^3}$$

is completely monotonic in $(0,\infty)$ if and only if $c \geq 1$ and its negative is completely monotonic in $(0,\infty)$ if and only if $c \leq 0$.

1. Introduction

A function f is said to be completely monotonic on an interval I if f has derivatives of all orders on I and

$$(-1)^n f^{(n)}(x) \geq 0 \tag{1}$$

for $x \in I$ and $n \geq 0$. The set of completely monotonic functions is denoted by $\mathcal{C}[I]$.

A positive function f is said to be logarithmically completely monotonic on an interval I if its logarithm $\ln f$ satisfies

$$(-1)^k [\ln f(x)]^{(k)} \geq 0 \tag{2}$$

for $k \in \mathbb{N}$ on I. The set of logarithmically completely monotonic functions is denoted by $\mathcal{L}[I]$.

A function f is called a Stieltjes transform if it can be of the form

$$f(x) = a + \int_0^\infty \frac{\mathrm{d}\,\mu(s)}{s+x}, \tag{3}$$

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where $a \ge 0$ and μ is a nonnegative measure on $[0,\infty)$ satisfying $\int_0^\infty \frac{1}{1+s}\,\mathrm{d}\,\mu(s) < \infty$. The set of Stieltjes transforms is denoted by $\mathcal{S}$.

To the best of our knowledge, the notion or terminology "logarithmically completely monotonic function" was introduced explicitly in [9], published formally in [8], and used immediately in [2, 4, 10, 11, 12]. Among other things, it is proved implicitly or explicitly in [2, 3, 8, 9, 10, 13] that $\mathcal{L}[I] \subset \mathcal{C}[I]$, but not conversely [9, 10]. Among other things, it is further revealed in [2, 13] that $\mathcal{S}\setminus\{0\} \subset \mathcal{L}[(0,\infty)] \subset \mathcal{C}[(0,\infty)]$. In [2, Theorem 1.1] and [4, 11] it is pointed out that the logarithmically completely monotonic functions on $(0,\infty)$ can be characterized as the infinitely divisible completely monotonic functions studied by Horn in [5, Theorem 4.4]. For more information on the logarithmically completely monotonic functions, please refer to [2, 4, 7, 10, 11, 13] and the references therein.

In [11, 12], it is proved that

$$\Phi(x) = \frac{[\Gamma(x+1)]^{1/x}}{x}\left(1+\frac{1}{x}\right)^x \in \mathcal{L}[(0,\infty)], \tag{4}$$

where $\Gamma(x)$ is the classical Euler gamma function defined by $\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}\,\mathrm{d}\,t$ for $\operatorname{Re} z > 0$, which is one of the most important special functions [1, 14, 15] and has much extensive applications in many branches, for example, statistics, physics, engineering, and other mathematical sciences. Motivated by [9, 12], among other things, the paper [2] proved that $\Phi(x) \in \mathcal{S}$ and $\ln\Phi(x) \in \mathcal{S}$ and the following explicit representations are obtained

$$\ln\Phi(x) = \int_0^\infty \frac{\phi(s)}{s+x}\,\mathrm{d}\,s \tag{5}$$

for $x > 0$, where

$$\phi(s) = \begin{cases} 1-s & \text{if } 0 \le s < 1 \\ 1-\dfrac{n}{s} & \text{if } n \le s < n+1 \text{ with } n \in \mathbb{N} \end{cases} \tag{6}$$

and

$$\Phi(x) = 1 + \int_0^\infty \frac{h(s)}{s+x}\,\mathrm{d}\,s \tag{7}$$

for $x > 0$ with

$$h(s) = \frac{s^{s-1}\sin(\pi\phi(s))}{\pi\,|1-s|^s\,|\Gamma(1-s)|^{1/s}} \tag{8}$$

for $s \ge 0$.

Define for $x \in (0,\infty)$

$$\Phi_c(x) = \frac{[\Gamma(x+1)]^{1/x}}{x^c}\left(1+\frac{1}{x}\right)^x. \tag{9}$$

It is clear that $\Phi_1(x) = \Phi(x)$.

The main purpose of this article is to confirm the range of c such that $\Phi_c(x) \in \mathcal{L}[(0,\infty)]$. Our main results are as follows.

Theorem 1. *The function*

$$\phi(x) = \psi''(x) + \frac{2 + (6+c)x + (4+3c)x^2 + (2+3c)x^3 + cx^4}{x^3(x+1)^3} \in \mathcal{C}[(0,\infty)] \quad (10)$$

if and only if $c \geq 1$ *and* $-\phi(x) \in \mathcal{C}[(0,\infty)]$ *if and only if* $c \leq 0$.

Theorem 2. *The function* $\Phi_c(x) \in \mathcal{L}[(0,\infty)]$ *if and only if* $c \geq 1$ *and* $[\Phi_c(x)]^{-1} \in \mathcal{L}[(0,\infty)]$ *if and only if* $c \leq 0$.

Remark 1. Since $\Phi_1(x)$ and $\ln \Phi_1(x)$ are both Stieltjes transforms, it is natural to ask whether the functions $\Phi_c(x)$ and $\ln \Phi_c(x)$ are Stieltjes transforms for $c \neq 1$.

2. Lemmas

In order to prove our main result, the following lemmas are necessary.

Lemma 1 ([1, 14, 15])**.** *For* $x > 0$ *and* $r > 0$,

$$\frac{1}{x^r} = \frac{1}{\Gamma(r)} \int_0^\infty t^{r-1} e^{-xt} \, \mathrm{d}t. \quad (11)$$

It is well known that the psi or digamma function is $\psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}$, the logarithmic derivative of the gamma function $\Gamma(x)$.

Lemma 2 ([1, 14, 15])**.** *The polygamma functions* $\psi^{(k)}(x)$ *can be expressed for* $x > 0$ *and* $k \in \mathbb{N}$ *as*

$$\psi^{(k)}(x) = (-1)^{k+1} k! \sum_{i=0}^{\infty} \frac{1}{(x+i)^{k+1}}, \quad (12)$$

$$\psi^{(k)}(x) = (-1)^{k+1} \int_0^\infty \frac{t^k e^{-xt}}{1-e^{-t}} \, \mathrm{d}t, \quad (13)$$

where $\gamma = 0.57721566\ldots$ *is the Euler-Mascheroni constant.*

For $i \in \mathbb{N}$,

$$\psi^{(i-1)}(x+1) = \psi^{(i-1)}(x) + \frac{(-1)^{i-1}(i-1)!}{x^i}. \quad (14)$$

Lemma 3 ([1, 14, 15])**.** *As* $x \to \infty$,

$$\ln \Gamma(x) = \left(x - \frac{1}{2}\right) \ln x - x + \frac{\ln(2\pi)}{2} + \frac{1}{12x} + O\left(\frac{1}{x}\right), \quad (15)$$

$$\psi(x) = \ln x - \frac{1}{2x} - \frac{1}{12x^2} + O\left(\frac{1}{x^2}\right), \quad (16)$$

$$(-1)^{n+1}\psi^{(n)}(x) = \frac{(n-1)!}{x^n} + \frac{n!}{2x^{n+1}} + \frac{(n+1)!}{12x^{n+2}} + O\left(\frac{1}{x^{n+2}}\right). \quad (17)$$

Lemma 4. *The function*

$$\varphi(t)=\frac{2e^{2t}-2(t+2)e^t+t^2+2t+2}{te^t(e^t-1)} \tag{18}$$

is strictly decreasing in $(0,\infty)$.

Proof. Straightforward computing yields

$$\begin{aligned}
\varphi'(t)&=\frac{2+2t+t^2+t^3-(6+4t+3t^2+2t^3)e^t+2(3+t+t^2)e^{2t}-2e^{3t}}{t^2e^t(e^t-1)^2}\\
&\triangleq\frac{\lambda_1(t)}{t^2e^t(e^t-1)^2},\\
\lambda_1'(t)&=2+2t+3t^2-(10+10t+9t^2+2t^3)e^t+2(7+4t+2t^2)e^{2t}-6e^{3t},\\
\lambda_1''(t)&=2+6t-(20+28t+15t^2+2t^3)e^t+4(9+6t+2t^2)e^{2t}-18e^{3t},\\
\lambda_1'''(t)&=6-54e^{3t}-(48+58t+21t^2+2t^3)e^t+16(6+4t+t^2)e^{2t},\\
\lambda_1^{(4)}(t)&=-\big[106+100t+27t^2+2t^3+162e^{2t}-32(8+5t+t^2)e^t\big]e^t\\
&\triangleq\lambda_2(t),\\
\lambda_2'(t)&=100+54t+6t^2-32(13+7t+t^2)e^t+324e^{2t},\\
\lambda_2''(t)&=6(9+2t)-32(20+9t+t^2)e^t+648e^{2t},\\
\lambda_2'''(t)&=4\big[3-8(29+11t+t^2)e^t+324e^{2t}\big],\\
\lambda_2^{(4)}(t)&=32(81e^t-t^2-13t-40)e^t.
\end{aligned}$$

It is clear that $\lambda_2^{(4)}(t)>0$ in $(0,\infty)$ and $\lambda_2^{(i)}(0)>0$ for $0\le i\le 3$. Therefore, the functions $\lambda_2^{(i)}(t)$ is increasing and positive for $0\le i\le 3$ in $(0,\infty)$. This implies that $\lambda_1^{(4)}(t)$ is negative in $(0,\infty)$. Since $\lambda_1^{(i)}(0)=0$ for $0\le i\le 3$, it follows that $\lambda_1^{(i)}(t)$ is decreasing and negative for $0\le i\le 3$ in $(0,\infty)$. This gives $\varphi'(t)<0$ in $(0,\infty)$. The proof of Lemma 4 is complete. □

3. Proofs of Theorem 1 and Theorem 2

Proof of Theorem 1. From formulas (11), (12) and (13), for $x\in(0,\infty)$ and any nonnegative integer i, it follows that

$$\begin{aligned}
\phi(x)&\triangleq\psi''(x)+g_2(x)+h_2(x)\\
&=\psi''(x)+\frac{2+cx-2x^2}{x^3}+\frac{2(3+3x+x^2)}{(x+1)^3}\\
&=\psi''(x)+\frac{2}{x^3}+\frac{c}{x^2}-\frac{2}{x}+\frac{2}{(1+x)^3}+\frac{2}{(1+x)^2}+\frac{2}{1+x}\\
&=\frac{c}{x^2}-\frac{2}{x}+\frac{2}{(1+x)^2}+\frac{2}{1+x}-2\sum_{i=2}^{\infty}\frac{1}{(x+i)^3}
\end{aligned}$$

$$
\begin{aligned}
&= \psi''(x+2) + \frac{c}{x^2} - \frac{2}{x} + \frac{2}{(1+x)^2} + \frac{2}{1+x} \\
&= c\int_0^\infty te^{-xt}\,\mathrm{d}t - 2\int_0^\infty e^{-xt}\,\mathrm{d}t + 2\int_0^\infty te^{-(x+1)t}\,\mathrm{d}t \\
&\quad + 2\int_0^\infty e^{-(x+1)t}\,\mathrm{d}t - \int_0^\infty \frac{t^2e^{-(x+2)t}}{1-e^{-t}}\,\mathrm{d}t \\
&= \int_0^\infty \bigl[(ct-2)e^{2t} + (2t-ct+4)e^t - (t^2+2t+2)\bigr]\frac{e^{-(x+2)t}}{1-e^{-t}}\,\mathrm{d}t \\
&\triangleq \int_0^\infty q(t)\frac{e^{-(x+2)t}}{1-e^{-t}}\,\mathrm{d}t
\end{aligned}
$$

and

$$\phi^{(i)}(x) = (-1)^i\int_0^\infty t^i q(t)\frac{e^{-(x+2)t}}{1-e^{-t}}\,\mathrm{d}t. \tag{19}$$

Standard argument shows that $q(t) \lesseqqgtr 0$ is equivalent to

$$c \lesseqqgtr \frac{2e^{2t} - 2(t+2)e^t + t^2 + 2t + 2}{te^t(e^t-1)} = \varphi(t) \tag{20}$$

for $t \geq 0$.

Using Lemma 4 and the fact that $\lim_{t\to 0}\varphi(t) = 1$ and $\lim_{t\to\infty}\varphi(t) = 0$ leads to $0 < \varphi(t) < 1$. If $c \geq 1$, then $q(t) \geq 0$; if $c \leq 0$, then $q(t) \leq 0$. This means that the function $\phi(x)$ is strictly completely monotonic in $(0,\infty)$ for $c \geq 1$ and $-\phi(x)$ is also strictly completely monotonic in $(0,\infty)$ for $c \leq 0$.

If $\phi(x)$ is completely monotonic in $(0,\infty)$, then by definition

$$\phi'(x) = \psi'''(x) - \frac{2(3+12x+17x^2+8x^3+3x^4)}{x^4(1+x)^4} - \frac{2c}{x^3} \leq 0 \tag{21}$$

which is equivalent to

$$c \geq \frac{x^3}{2}\left(\psi'''(x) - \frac{2(3+12x+17x^2+8x^3+3x^4)}{x^4(1+x)^4}\right) \to 1 \tag{22}$$

as $x \to \infty$ by using the asymptotic formula (17). Similarly, it is easy to see that the necessary condition of $-\phi(x)$ being completely monotonic in $(0,\infty)$ is $c \leq 0$. The proof of Theorem 1 is complete. □

The first proof of Theorem 2. Taking logarithm of $\Phi_c(x)$ gives

$$\ln\Phi_c(x) = x\ln\left(1+\frac{1}{x}\right) + \frac{\ln\Gamma(x+1)}{x} - c\ln x.$$

Differentiating yields

$$[\ln\Phi_c(x)]' = \ln\left(1+\frac{1}{x}\right) - \frac{1}{x+1} + \frac{x\psi(x+1) - \ln\Gamma(x+1)}{x^2} - \frac{c}{x} \tag{23}$$

and

$$[\ln\Phi_c(x)]^{(n)} = (-1)^{(n-1)}(n-1)!x\left[\frac{1}{(x+1)^n} - \frac{1}{x^n}\right]$$

$$
\begin{aligned}
&+(-1)^n(n-2)!n\left[\frac{1}{(x+1)^{n-1}}-\frac{1}{x^{n-1}}\right]\\
&+\frac{h_n(x)}{x^{n+1}}+(-1)^n(n-1)!\frac{c}{x^n}\\
&=(-1)^n(n-2)!\left[\frac{c(n-1)-x}{x^n}+\frac{x+n}{(x+1)^n}\right]+\frac{h_n(x)}{x^{n+1}},
\end{aligned}
$$

where $n\ge 2$, $\psi^{(-1)}(x+1)=\ln\Gamma(x+1)$, $\psi^{(0)}(x+1)=\psi(x+1)$, and

$$
h_n(x)=\sum_{k=0}^{n}\frac{(-1)^{n-k}n!x^k\psi^{(k-1)}(x+1)}{k!}, \tag{24}
$$

$$
h_n'(x)=x^n\psi^{(n)}(x+1)\begin{cases}>0 & \text{if } n \text{ is odd},\\ <0 & \text{if } n \text{ is even}.\end{cases} \tag{25}
$$

Therefore, we have

$$
\begin{aligned}
&(-1)^n x^{n+1}[\ln\Phi_c(x)]^{(n)}+(-1)^{n+1}h_n(x)\\
&=(n-2)!\left\{c(n-1)-x+\frac{x^n(x+n)}{(x+1)^n}\right\}x
\end{aligned}
$$

and, by (14),

$$
\begin{aligned}
&\frac{\mathrm{d}\left\{(-1)^n x^{n+1}[\ln\Phi_c(x)]^{(n)}\right\}}{\mathrm{d}x}\\
&=(-1)^n x^n\psi^{(n)}(x+1)+(n-2)!\left\{c(n-1)-2x\right.\\
&\quad\left.+\frac{x^n[n+n^2+(2+2n)x+2x^2]}{(x+1)^{n+1}}\right\}\\
&=x^n\left\{(-1)^n\psi^{(n)}(x+1)+(n-2)!\left[\frac{c(n-1)-2x}{x^n}\right.\right.\\
&\quad\left.\left.+\frac{n+n^2+(2+2n)x+2x^2}{(x+1)^{n+1}}\right]\right\}\\
&=x^n\left\{(-1)^n\psi^{(n)}(x)+\frac{n!}{x^{n+1}}+(n-2)!\left[\frac{c(n-1)-2x}{x^n}\right.\right.\\
&\quad\left.\left.+\frac{n+n^2+(2+2n)x+2x^2}{(x+1)^{n+1}}\right]\right\}\\
&=x^n\left\{(-1)^n\psi^{(n)}(x)+\frac{n!}{x^{n+1}}+(n-2)!\left[\frac{c(n-1)-2x}{x^n}\right.\right.\\
&\quad\left.\left.+\frac{n(n+1)+2(n+1)x+2x^2}{(x+1)^{n+1}}\right]\right\}\\
&=x^n\left\{(-1)^n\psi^{(n)}(x)+(n-2)!\left[\frac{n(n-1)+c(n-1)x-2x^2}{x^{n+1}}\right.\right.\\
&\quad\left.\left.+\frac{n(n+1)+2(n+1)x+2x^2}{(x+1)^{n+1}}\right]\right\}\\
&\triangleq x^n\left\{(-1)^n\psi^{(n)}(x)+(n-2)![g_n(x)+h_n(x)]\right\}
\end{aligned}
$$

with

$$g_n'(x) = -(n-1)g_{n+1}(x) \quad \text{and} \quad h_n'(x) = -(n-1)h_{n+1}(x)$$

which implies

$$g_2^{(n-2)}(x) = (-1)^n(n-2)!g_n(x)$$

and

$$h_2^{(n-2)}(x) = (-1)^n(n-2)!h_n(x)$$

by induction. Hence, by using Theorem 1, we have

$$\frac{\mathrm{d}\{(-1)^n x^{n+1}[\ln\Phi_c(x)]^{(n)}\}}{\mathrm{d}x} = (-1)^n x^n \phi^{(n-2)}(x) \begin{cases} > 0 & \text{if and only if } c \ge 1, \\ < 0 & \text{if and only if } c \le 0, \end{cases}$$

and the function $(-1)^n x^{n+1}[\ln\Phi_c(x)]^{(n)}$ is increasing (or decreasing) if and only if $c \ge 1$ (or $c \le 0$) in $(0,\infty)$. From

$$\lim_{x\to 0}\{(-1)^n x^{n+1}[\ln\Phi_c(x)]^{(n)}\} = 0,$$

it is deduced that

$$(-1)^n x^{n+1}[\ln\Phi_c(x)]^{(n)} \begin{cases} > 0 & \text{if and only if } c \ge 1 \\ < 0 & \text{if and only if } c \le 0 \end{cases}$$

and

$$(-1)^n[\ln\Phi_c(x)]^{(n)} \begin{cases} > 0 & \text{if and only if } c \ge 1 \\ < 0 & \text{if and only if } c \le 0 \end{cases}$$

for $n \ge 2$ in $(0,\infty)$. This implies the function $[\ln\Phi_c(x)]'$ is increasing (or decreasing) if and only if $c \ge 1$ (or $c \le 0$) in $(0,\infty)$. It is ready to obtain $\lim_{x\to\infty}[\ln\Phi_c(x)]' = 0$, so

$$[\ln\Phi_c(x)]' \begin{cases} < 0 & \text{if and only if } c \ge 1 \\ > 0 & \text{if and only if } c \le 0 \end{cases}$$

and $\ln\Phi_c(x)$ is decreasing (or increasing) if and only if $c \ge 1$ (or $c \le 0$) in $(0,\infty)$. The first proof of Theorem 2 is complete. □

The second proof of Theorem 2. Write

$$\Phi_c(x) = \frac{1}{x^{c-1}}\Phi(x).$$

Hence

$$f(x) \equiv \ln[\Phi_c(x)] = -(c-1)\ln x + \ln[\Phi(x)].$$

By applying one of the results in [11] that $\Phi(x)$ is logarithmically completely monotonic in $(0,\infty)$, it is easy to show $(-1)^n f^{(n)}(x) \ge 0$ in $(0,\infty)$ for all $n \in \mathbb{N}$ if $c \ge 1$.

For the part of $c < 1$, the second part of Theorem 2 is proved if one uses

$$\ln \frac{1}{\Phi_c(x)} = -\ln(\Phi_c(x)).$$

If the function $\Phi_c(x)$ is logarithmically completely monotonic in $(0,\infty)$, then by definition $[\ln \Phi_c(x)]' \le 0$ which is equivalent to

$$c \ge x \ln\left(1 + \frac{1}{x}\right) - \frac{x}{x+1} + \frac{x\psi(x+1) - \ln\Gamma(x+1)}{x} \triangleq \vartheta(x) \tag{26}$$

from (23). If $\frac{1}{\Phi_c(x)}$ is logarithmically completely monotonic in $(0,\infty)$, then by definition $[\ln \Phi_c(x)]' \ge 0$ which is equivalent to the reversed inequality of (26). By L'Hospital rule, it is easy to obtain that $\lim_{x\to 0} \vartheta(x) = 0$. Utilizing directly Lemma 3 yields $\lim_{x\to\infty} \vartheta(x) = 1$. Therefore, the necessary condition of $\Phi_c(x)$ being logarithmically completely monotonic in $(0,\infty)$ is $c \ge 1$ and the necessary condition of $\frac{1}{\Phi_c(x)}$ being logarithmically completely monotonic in $(0,\infty)$ is $c \le 0$. The second proof of Theorem 2 is complete. □

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