A SUBADDITIVE PROPERTY OF THE DIGAMMA FUNCTION

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Abstract. We prove that the function $x \mapsto \psi(a+e^x)(a>0)$ is subadditive on $(-\infty,+\infty)$ if and only if $a \geq c_0$, where c_0 is the only positive zero of $\psi(x)$.

1. Introduction

Euler's gamma function for $x>0$ is defined as

$$\Gamma(x)=\int_0^\infty t^x e^{-t}\frac{dt}{t}.$$

The digamma (or psi) function $\psi(x)$ is defined as the logarithmic derivative of $\Gamma(x)$. It has the following series representation(see, for example, [7, (1.8)]):

$$\psi(x)=-\gamma+\sum_{n=0}^{\infty}(\frac{1}{n+1}-\frac{1}{x+n}). \tag{1.1}$$

Here $\gamma=0.57721\ldots$ denotes Euler's constant. We further note the following asymptotic expression(see, for example, [2, (1.4)]:

$$\psi(x)=\log x+O(\frac{1}{x}). \tag{1.2}$$

There exist many inequalities for the gamma and digamma functions. For the recent developments in this area, we refer the reader to the articles [1]-[3], [7], [8] and the references therein.

A function f defined on a set D of real numbers is said to be subadditive on D if $f(x+y)\leq f(x)+f(y)$ for all $x,y\in D$ such that $x+y\in D$. If instead $f(xy)\leq f(x)f(y)$ for all $x,y\in D$ such that $xy\in D$, then $f(x)$ is said to be submultiplicative on D.

Inspired by (1.2) and a result of Gustavsson et al. [6], which asserts that if $a\geq 1$, then the function $f(x)=\log(a+x)$ is submultiplicative on $[0,+\infty)$ if and only if $a\geq e$, Alzer and Ruehr[3] proved the following submultiplicative property of the digamma function:

Theorem 1.1. *Let $a>0$ be fixed. Then $\psi(a+x)$ as a function of x is submultiplicative on $[0,+\infty)$ if and only if $a\geq a_0=3.203171\ldots$. (Here, a_0 denotes the only positive real number which satisfies $\psi(a_0)=1$.)*

The above result asserts that for $a\geq a_0, x,y\geq 0$,

$$\psi(a+xy)\leq\psi(a+x)\psi(a+y). \tag{1.3}$$

We note here[5] that for $a>0, x,y\geq 0$,

$$\psi(a)+\psi(a+x+y)\leq\psi(a+x)+\psi(a+y).$$

As $\psi(x)$ is an increasing function(see [5]), we obtain from the above that

$$\psi(a)+\psi(a+xy)\leq\psi(a+x)+\psi(a+y), \tag{1.4}$$

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for $a > 0, x, y \geq 0$, provided that $xy \leq x + y$ or $1 \leq 1/x + 1/y$. In particular, when $x \leq 1$ or $y \leq 1$, inequality (1.4) is valid and this would then give a refinement of (1.3) when $a \geq a_0$ since we now have

$$1 + \psi(a + xy) \leq \psi(a) + \psi(a + xy) \leq \psi(a + x) + \psi(a + y),$$

and that

$$\psi(a + x) + \psi(a + y) - 1 \leq \psi(a + x)\psi(a + y).$$

We point out here (1.4) does not hold in general. In fact, it follows from (1.2) and (1.4) that when $x \to +\infty$,

$$\psi(a) + \log y \leq \psi(a + y),$$

and the above inequality does not hold for $y > 1$ if we take $a \to +\infty$, in view of (1.2) again. Similarly, it is not possible to compare $\psi(a)\psi(a + xy)$ and $\psi(a + x)\psi(a + y)$ for $a \geq a_0, x, y \geq 0$ in general. Certainly when we fix a and set $x = y \to +\infty$, we will have by (1.2),

$$\psi(a)\psi(a + xy) < \psi(a + x)\psi(a + y),$$

and the above inequality is reversed when we set $x = y = a \to +\infty$.

It is now interesting to ask whether the following inequality holds for all $x, y \geq 0$:

$$\psi(a + xy) \leq \psi(a + x) + \psi(a + y), \tag{1.5}$$

for a larger than some constant. In view of (1.2), this can be thought as an analogue of the property $\log(xy) = \log x + \log y$, $x, y > 0$ and the easily checked fact: $\log(a + xy) \leq \log(a + x) + \log(a + y)$, $a \geq 1, x, y \geq 0$. We may also interpret (1.5) as asserting that the function $x \mapsto \psi(a + e^x)$ is subadditive on $(-\infty, +\infty)$ for a larger than some constant.

By taking $x = y = 0$ in (1.5), we see that it is necessary to have $a \geq c_0$, where c_0 is the only positive zero of $\psi(x)$ and it is our goal in this paper to prove that this condition is also sufficient for (1.5) to hold.

2. A Lemma

Lemma 2.1. *The function $f(x) = x\psi'(a + x)$ is strictly increasing on $[0, +\infty)$ for $a \geq 1$.*

Proof. We have

$$f'(x) = \psi'(a + x) + x\psi''(a + x).$$

We now use a method in [4] to show that $f'(x) > 0$ for $x \geq 0$. First note the following two asymptotic expressions([4, p. 2668]):

$$\psi'(x) = \frac{1}{x} + O(\frac{1}{x^2}), \psi''(x) = -\frac{1}{x^2} + O(\frac{1}{x^3}).$$

It follows that

$$\lim_{x \to +\infty} f'(x) = 0.$$

Hence it suffices to show that $f(x) - f(x + 1) > 0$. Using (1.1) we get

$$\begin{aligned} f(x) - f(x+1) &= -\frac{1}{(x+a)^2} + \frac{2a}{(x+a)^3} + \sum_{n=1}^{\infty} \frac{2}{(n+x+a)^3} \\ &\geq -\frac{1}{(x+a)^2} + \sum_{n=0}^{\infty} \frac{2}{(n+x+a)^3} \\ &> -\frac{1}{(x+a)^2} + \sum_{n=0}^{\infty} (\frac{1}{(n+x+a)^2} - \frac{1}{(n+x+a+1)^2}) = 0, \end{aligned}$$

where the last inequality follows from the inequality

$$\frac{2}{u^3} > \frac{1}{u^2} - \frac{1}{(1+u)^2}, \quad u > 0.$$

The proof is now complete. □

3. Main Result

Theorem 3.1. *The function $x \mapsto \psi(a + e^x)(a > 0)$ is subadditive on $(-\infty, +\infty)$ if and only if $a \geq c_0$, where c_0 is the only positive zero of $\psi(x)$.*

Proof. From our discussions above, it suffices to show (1.5) holds for all $x, y \geq 0$ when $a \geq c_0$. If $x \leq 1$ or $y \leq 1$, then (1.5) follows from (1.4). Hence from now on we may assume that $x, y \geq 1$. In this case we define

$$f(x, y) = \psi(a + x) + \psi(a + y) - \psi(a + xy),$$

and note that $f(x, 1) = \psi(a + 1) > 0$. If $y > 1$, then as $c_0 > 1$(since $\psi(1) = -\gamma$ by (1.1)), Lemma 2.1 implies that $\partial f/\partial x < 0$ so that by (1.2),

$$f(x, y) > \lim_{u \to +\infty} f(u, y) = \psi(a + y) - \log y.$$

Now that we have[1, (2.2)] for $x > 0$,

$$\frac{1}{2x} < \log x - \psi(x) < \frac{1}{x}.$$

It follows from this and the mean value theorem that

$$\psi(a + y) - \log y > \log(a + y) - \log y - \frac{1}{a + y} > \frac{a}{a + y} - \frac{1}{a + y} \geq 0.$$

From this we conclude that $f(x, y) \geq 0$ for $x, y \geq 0$ and this completes the proof. □

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