TRAPEZOIDAL TYPE INEQUALITIES FOR $n-$TIME DIFFERENTIABLE FUNCTIONS

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ABSTRACT. In this paper, by utilising a result given by Fink [2], we obtain some new results relating to the trapezoidal inequality and some of its generalisations for $n-$time differentiable functions.

1. Introduction

The following result is known in the literature as Ostrowski's inequality [3, p. 468].

Theorem 1. *Let $f : [a, b] \to \mathbb{R}$ be a differentiable mapping on (a, b) with the property that $|f'(t)| \le M$ for all $t \in (a, b)$. Then*

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(t)\, dt \right| \le \left[\frac{1}{4} + \frac{\left(x - \frac{a+b}{2} \right)^2}{(b-a)^2} \right] (b-a) M.$$

The Ostrowski inequality has been generalised in a number of different ways, see [3] and [1].

Fink [2] also obtained the following result for $n-$time differentiable functions.

Theorem 2. *Let $f^{(n-1)}(t)$ be absolutely continuous on $[a, b]$ with $f^{(n)}(t) \in L_p(a, b)$ and let*

$$F_k(x) := \frac{n-k}{k!} \left[\frac{f^{(k-1)}(a)(x-a)^k - f^{(k-1)}(b)(x-b)^k}{b-a} \right], \quad k = 1, 2, \ldots, n-1; \tag{1.1}$$

then

$$\left| \frac{1}{n} \left[f(x) + \sum_{k=1}^{n-1} F_k(x) \right] - \frac{1}{b-a} \int_a^b f(y)\, dy \right| \tag{1.2}$$

$$\le \begin{cases} \frac{\left[(x-a)^{nq+1} + (b-x)^{nq+1} \right]^{\frac{1}{q}}}{n!(b-a)} B^{\frac{1}{q}}(q+1, (n-1)q+1) \left\| f^{(n)} \right\|_p, & \text{for } f \in L_p[a, b]; \\ & p > 1,\ \frac{1}{p} + \frac{1}{q} = 1; \\ \frac{(n-1)^{n-1}}{n^n n!} \cdot \frac{\max\{(x-a)^n, (b-x)^n\}}{b-a} \left\| f^{(n)} \right\|_1; & \text{for } f \in L_1[a, b]; \\ \left[\frac{(x-a)^{n+1} + (b-x)^{n+1}}{n(n+1)!(b-a)} \right] \left\| f^{(n)} \right\|_\infty; & \text{for } f \in L_\infty[a, b]; \end{cases}$$

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where $B(\alpha, \beta)$ is Euler's Beta function,

$$\left\|f^{(n)}\right\|_p := \left(\int_a^b \left|f^{(n)}(t)\right|^p dt\right)^{\frac{1}{p}}, \quad p \geq 1 \quad \text{and}$$

$$\left\|f^{(n)}\right\|_\infty := ess\sup\left|f^{(n)}(t)\right| < \infty. \tag{1.3}$$

Remark 1. *The result above on the infinite norm was given by Milovanović and Pečarić in 1976 (see* [3, p. 468]*).*

Note that for $n = 1$ and the infinite norm, Theorem 2 reduces to Theorem 1.

In the next section we develop an integral equality that will permit us to obtain bounds for the error estimate in a generalised trapezoid formula. The new results complement some of the earlier inequalities related to the trapezoidal rule reported in [1]

2. The Results

The following generalisation of the trapezoid formula holds:

Theorem 3. *Let $f : [a, b] \to \mathbb{R}$ be a mapping such that its $(n-1)^{th}$ derivative $f^{(n-1)}$ is absolutely continuous on $[a, b]$. Then we have the equality*

$$\begin{aligned} &\frac{1}{n}\left[\frac{f(a)+f(b)}{2} + \sum_{k=1}^{n-1} \frac{(n-k)(b-a)^{k-1}}{k!} \right. \\ &\quad \left. \times \left\{\frac{f^{(k-1)}(a) + (-1)^{k-1} f^{(k-1)}(b)}{2}\right\}\right] - \frac{1}{b-a}\int_a^b f(y)\,dy \\ &= \frac{1}{2n!(b-a)}\int_a^b (t-a)(b-t)\left[(b-t)^{n-2} + (-1)^{n-2}(t-a)^{n-2}\right] f^{(n)}(t)\,dt. \end{aligned} \tag{2.1}$$

Proof. We may use Fink's identity [2] which states

$$\begin{aligned} &\frac{1}{n}\left[f(x) + \sum_{k=1}^{n-1} F_k(x)\right] - \frac{1}{b-a}\int_a^b f(y)\,dy \\ &\qquad = \frac{1}{n!(b-a)}\int_a^b (x-t)^{n-1} K(t,x) f^{(n)}(t)\,dt, \end{aligned} \tag{2.2}$$

where

$$K(t,x) := \begin{cases} t-a & \text{if } a \leq t \leq x \leq b \\ t-b & \text{if } a \leq x \leq t \leq b \end{cases}$$

and $F_k(x)$ is defined by (1.1).

If in (2.2) we put $x = a$, then we obtain

$$\begin{aligned} &\frac{1}{n}\left[f(a) + \sum_{k=1}^{n-1} F_k(a)\right] - \frac{1}{b-a}\int_a^b f(y)\,dy \\ &\qquad = \frac{(-1)^n}{n!(b-a)}\int_a^b (t-a)^{n-1}(b-t) f^{(n)}(t)\,dt, \end{aligned} \tag{2.3}$$

where

$$F_k(a) = \frac{(-1)^{k-1}(n-k) f^{(k-1)}(b)(b-a)^{k-1}}{k!}.$$

Similarly, if in (2.2) we put $x = b$, we get

$$\frac{1}{n}\left[f(b) + \sum_{k=1}^{n-1} F_k(b)\right] - \frac{1}{b-a}\int_a^b f(y)\,dy = \frac{1}{n!(b-a)}\int_a^b (b-t)^{n-1}(t-a) f^{(n)}(t)\,dt, \tag{2.4}$$

where

$$F_k(b) = \frac{(n-k) f^{(k-1)}(a)(b-a)^{k-1}}{k!}.$$

Adding (2.3) to (2.4) and dividing by 2, we have

$$\frac{1}{2n}\left[f(a) + f(b) + \sum_{k=1}^{n-1}\{F_k(a) + F_k(b)\}\right] - \frac{1}{b-a}\int_a^b f(y)\,dy$$
$$= \frac{1}{2n!(b-a)}\int_a^b (t-a)(b-t)\left[(b-t)^{n-2} + (-1)^{n-2}(t-a)^{n-2}\right] f^{(n)}(t)\,dt,$$

replacing $F_k(a)$ and $F_k(b)$ we obtain identity (2.1), hence the theorem is proved. ∎

Remark 2. *(a) For $n = 1$, we recapture the known identity,*

$$\frac{f(a) + f(b)}{2} - \frac{1}{b-a}\int_a^b f(y)\,dy = \frac{1}{b-a}\int_a^b \left[t - \left(\frac{a+b}{2}\right)\right] f'(t)\,dt. \tag{2.5}$$

(b) For $n = 2$, we deduce the equality below, which is also well known in the literature,

$$\frac{f(a) + f(b)}{2} - \frac{1}{b-a}\int_a^b f(y)\,dy = \frac{1}{2(b-a)}\int_a^b (t-a)(b-t) f''(t) \tag{2.6}$$
$$= \frac{1}{2(b-a)}\int_a^b \left[\left(\frac{b-a}{2}\right)^2 - \left(t - \left(\frac{a+b}{2}\right)\right)^2\right] f''(t)\,dt.$$

(c) For $n = 3$, we have some extra terms involving the first derivative at the end points, namely:

$$\frac{f(a) + f(b)}{2} + \frac{b-a}{12}\left[f'(a) - f'(b)\right] - \frac{1}{b-a}\int_a^b f(y)\,dy \tag{2.7}$$
$$= \frac{1}{6(b-a)}\int_a^b (t-a)(b-t)\left(\frac{a+b}{2} - t\right) f'''(t)\,dt.$$

(d) *Finally, for* $n = 4$, *we have the following*

$$\frac{f(a)+f(b)}{2}+\frac{b-a}{8}\left[f'(a)-f'(b)\right]+\frac{(b-a)^2}{48}\left[f''(a)+f''(b)\right]-\frac{1}{b-a}\int_a^b f(y)\,dy \tag{2.8}$$
$$=\frac{1}{48(b-a)}\int_a^b (t-a)(b-t)\left[(b-t)^2+(t-a)^2\right]f^{(4)}(t)\,dt$$
$$=\frac{1}{24(b-a)}\int_a^b (t-a)(b-t)\left[\left(t-\frac{a+b}{2}\right)^2+\left(\frac{b-a}{2}\right)^2\right]f^{(4)}(t)\,dt.$$

The following inequalities can now be stated.

Theorem 4. *Let* $f:[a,b]\to\mathbb{R}$ *be a mapping such that* $f^{(n-1)}$ *is absolutely continuous on* $[a,b]$. *Define*

$$T(a,b,n) := \frac{1}{n}\left[\frac{f(a)+f(b)}{2}+\sum_{k=1}^{n-1}\frac{(n-k)(b-a)^{k-1}}{k!}\left\{\frac{f^{(k-1)}(a)+(-1)^{k-1}f^{(k-1)}(b)}{2}\right\}\right]-\frac{1}{b-a}\int_a^b f(y)\,dy,$$

then

$$|T(a,b,n)| \le \begin{cases} \frac{(b-a)^{n-1+\frac{1}{q}}}{n!}B^{\frac{1}{q}}\left(q+1,(n-1)q+1\right)\left\|f^{(n)}\right\|_p & \text{for } \frac{1}{p}+\frac{1}{q}=1;\ p>1,\ q>1; \\ \frac{(b-a)^{n-\frac{1}{2}}}{\sqrt{2(2n+1)!n!}}\left[2(2n-2)!+(-1)^n(n!)^2\right]^{\frac{1}{2}}\left\|f^{(n)}\right\|_2 & \text{for } p=2, q=2; \\ \frac{(b-a)^n}{n(n+1)!}\left\|f^{(n)}\right\|_\infty, \end{cases} \tag{2.9}$$

where $B(x,y)$ *is the Beta function and* $\left\|f^{(n)}\right\|_p$, $\left\|f^{(n)}\right\|_\infty$ *are as defined in (1.3).*

Proof. From (2.1) and the definition of $T(a,b,n)$ we have

$$|T(a,b,n)| = \left|\frac{1}{2n!(b-a)}\int_a^b Y(t)\,f^{(n)}(t)\,dt\right|, \tag{2.10}$$

where

$$Y(t) := (t-a)(b-t)^{n-1}+(-1)^n(t-a)^{n-1}(b-t). \tag{2.11}$$

By Hölder's inequality

$$|T(a,b,n)| \le \frac{1}{2n!(b-a)}\left(\int_a^b |Y(t)|^q\,dt\right)^{\frac{1}{q}}\left(\int_a^b \left|f^{(n)}(t)\right|^p dt\right)^{\frac{1}{p}}, \quad \frac{1}{p}+\frac{1}{q}=1,\ q>1,\ p>1. \tag{2.12}$$

Now, let us provide some upper bounds for the integral: $\int_a^b |Y(t)|^q \, dt$. For real α and β and $q > 1$, we have the elementary inequality

$$|\alpha + \beta|^q \leq 2^{q-1} \left(|\alpha|^q + |\beta|^q \right). \tag{2.13}$$

Utilising (2.13), we obtain

$$\begin{aligned} \int_a^b |Y(t)|^q \, dt &\leq 2^{q-1} \int_a^b \left[(t-a)^q (b-t)^{(n-1)q} + (t-a)^{(n-1)q} (b-t)^q \right] dt \\ &= 2^q (b-a)^{nq+1} B(q+1, (n-1)q+1) \end{aligned}$$

by the substitution $w = \frac{t-a}{b-a}$ and the symmetry of the Beta function,

$$B(\alpha, \beta) = B(\beta, \alpha) = \int_0^1 t^{\alpha-1} (1-t)^{\beta-1} \, dt; \quad \alpha, \beta > 0.$$

Also

$$\left(\int_a^b |Y(t)|^q \, dt \right)^{\frac{1}{q}} \leq 2 (b-a)^{n+\frac{1}{q}} B^{\frac{1}{q}} (q+1, (n-1)q+1)$$

so from (2.12) we deduce

$$|T(a,b,x)| \leq \frac{(b-a)^{n-1+\frac{1}{q}}}{n!} B^{\frac{1}{q}} (q+1, (n-1)q+1) \left\| f^{(n)} \right\|_p$$

and the first part of the theorem is proved.

For the Euclidean norm ($p = 2$, $q = 2$), we can compute exactly

$$\begin{aligned} &\int_a^b |Y(t)|^2 \, dt \\ &= \int_a^b \left| (t-a)(b-t)^{n-1} + (-1)^n (t-a)^{n-1} (b-t) \right|^2 dt \\ &= \int_a^b \left((t-a)^2 (b-t)^{2n-2} + 2(-1)^n (t-a)^n (b-t)^n + (t-a)^{2n-2} (b-t)^2 \right) dt \\ &= 2(b-a)^{2n+1} \left[B(3, 2n-1) + (-1)^n B(n+1, n+1) \right] \\ &= \frac{2(b-a)^{2n+1}}{(2n+1)!} \left[2(2n-2)! + (-1)^n (n!)^2 \right]. \end{aligned}$$

From (2.12)

$$|T(a,b,n)| \leq \frac{(b-a)^{n-\frac{1}{2}}}{\sqrt{2(2n+1)!n!}} \left[2(2n-2)! + (-1)^n (n!)^2 \right]^{\frac{1}{2}} \left\| f^{(n)} \right\|_2$$

and the second part of the theorem is proved.

For the third part of the theorem, observe that

$$\begin{aligned} \int_a^b |Y(t)| \, dt &= \int_a^b \left| (t-a)(b-t)^{n-1} + (-1)^n (t-a)^{n-1} (b-t) \right| dt \\ &\leq \int_a^b (t-a)(b-t)^{n-1} \, dt + \int_a^b (t-a)^{n-1} (b-t) \, dt \\ &= \frac{2(b-a)^{n+1}}{n(n+1)}. \end{aligned}$$

From (2.12)

$$|T(a,b,n)| \leq \frac{(b-a)^n}{n(n+1)!} \left\| f^{(n)} \right\|_\infty ,$$

the third part of the theorem is proved, hence Theorem 4 is proved. ∎

For the 1−norm we can also delineate the following theorem.

Theorem 5. *Let $f:[a,b] \to \mathbb{R}$ be a mapping such that $f^{(n-1)}$ is absolutely continuous on $[a,b]$.*

(i) *For n even, let $n=2k$, $k \geq 1$, then*

$$|T(a,b,2k)| \leq \begin{cases} \max\limits_{t^*} \left\{ \frac{(t^*-a)(b-t^*)^{2k-1}+(t^*-a)^{2k-1}(b-t^*)}{2(2k)!(b-a)} \right\} \left\| f^{(2k)} \right\|_1 , & \text{for } k>2; \\ \frac{b-a}{8} \left\| f'' \right\|_1 & \text{for } k=1; \\ \frac{(b-a)^3}{384} \left\| f^{(4)} \right\|_1 & \text{for } k=2; \end{cases}$$

where $t^* \in (a, \frac{a+b}{2}]$ is a solution of the polynomial equation

$$(b-t)^{2k-1} - (t-a)^{2k-1} + (2k-1)\left[(t-a)^{2k-2}(b-t) - (t-a)(b-t)^{2k-2} \right] = 0.$$

(ii) *For n odd, let $n=2k+1$, $k \geq 0$*

$$|T(a,b,2k+1)| \leq \begin{cases} \max\limits_{\tau} \left\{ \frac{(\tau-a)(b-\tau)^{2k}-(\tau-a)^{2k}(b-\tau)}{2(2k+1)!(b-a)} \right\} \left\| f^{(2k)} \right\|_1 & \text{for } k \geq 1 \\ \frac{1}{2} \left\| f' \right\|_1 \quad \text{for} \quad k=0, \ n=1 \end{cases},$$

where $\tau \in (a, \frac{a+b}{2}]$ is a solution of the polynomial equation

$$(t-a)^{2k} + (b-t)^{2k} - 2k\left[(t-a)(b-t)^{2k-1} + (t-a)^{2k-1}(b-t) \right] = 0.$$

The following two lemmas will be useful in the proof of Theorem 5.

Lemma 1. *Let $b>a$, k is an integer and $t \in [a,b]$. Define $M(t,k) := (t-a)(b-t)^{2k-1} + (t-a)^{2k-1}(b-t)$, then $M(t,k)$ has exactly two zeros as a function of t on $[a,b]$.*

Proof. Observe that

$$M(t,k) = -(t-a)(t-b)\left[(t-a)^{2k-2} + (t-b)^{2k-2} \right].$$

Since for $t \in (a,b)$, $\left[(t-a)^{k-1} \right]^2 + \left[(t-b)^{k-1} \right]^2 > 0$, hence $M(t,k)=0$ has the only real solutions $t=a$ and $t=b$. ∎

Lemma 2. *Let $b>a$, k is an integer and $t \in [a,b]$. Define*

$$P(t,k) = (t-a)(b-t)^{2k} + (t-a)^{2k}(b-t),$$

then $P(t,k)$ has exactly three zeros in $[a,b]$.

Proof. We have

$$P(t,k) = (t-a)(t-b)\left[(t-b)^{2k-1} + (t-b)^{2k-1}\right].$$

Observe that

$$P(a,k) = P(b,k) = 0.$$

Since

$$(t-a)^{2k-1} + (t-b)^{2k-1} = (2t-a+b)\left[(t-a)^{2k-2} + \cdots + (b-t)^{2k-2}\right]$$

and

$$(t-a)^{2k-2} + (t-a)^{2k-3}(b-t)\cdots + (t-a)(b-t)^{2k-3} + (b-t)^{2k-2} > 0$$

for $t \in [a,b]$, hence the third solution of $P(t,k) = 0$ for $t \in [a,b]$ is $t = \frac{a+b}{2}$. ∎

Proof of Theorem 5. (i) Consider the case of n even, let $n = 2k$, $k \geq 1$ and denote

$$Y := \sup_{t\in[a,b]} \left|(t-a)(b-t)^{2k-1} + (t-a)^{2k-1}(b-t)\right|.$$

For $M(t,k)$ defined above, simple calculations show that

$$M'(t,k) = (b-t)^{2k-1} - (t-a)^{2k-1} + (2k-1)\left[(t-a)^{2k-2}(b-t) - (t-a)(b-t)^{2k-2}\right],$$

$$M''(t,k) = -2(2k-1)\left[(t-a)^{2k-2} + (b-t)^{2k-2}\right] + (2k-2)(2k-1)\left[(t-a)(b-t)^{2k-3} + (t-a)^{2k-3}(b-t)\right],$$

and

$$M(a,k) = M(b,k) = 0; \quad M\left(\frac{a+b}{2},k\right) = 2\left(\frac{b-a}{2}\right)^{2k-2} > 0$$

$$M'(a,k) = (b-a)^{2k-1} > 0; \quad M'(b,k) = -(b-a)^{2k-1} < 0, \quad M'\left(\frac{a+b}{2},k\right) = 0,$$

$$M''\left(\frac{a+b}{2},k\right) = 2(2k-1)(2k-2)\left(\frac{b-a}{2}\right)^{2k-2} > 0 \quad \text{for} \quad k > 2.$$

The local extrema for the function $M(\cdot,k)$ are the real numbers $t^* \in \left[a, \frac{a+b}{2}\right]$ that are solutions of the polynomial equation

$$(b-t)^{2k-1} - (t-a)^{2k-1} + (2k-1)\left[(t-a)^{2k-2}(b-t) - (t-a)(b-t)^{2k-2}\right] = 0.$$

Therefore, by Lemma 1

$$Y = \max_{t^*}\left\{(t^*-a)(b-t^*)^{2k-1} + (t^*-a)^{2k-1}(b-t^*)\right\} > 0$$

hence

$$|T(a,b,2k)| \leq \max_{t^*}\left\{\frac{(t^*-a)(b-t^*)^{2k-1} + (t^*-a)^{2k-1}(b-t^*)}{2(2k)!(b-a)}\right\}\left\|f^{(2k)}\right\|_1$$

for $k > 2$.

For the two special cases $k=1$, $(n=2)$, we have

$$|T(a,b,2)| \leq \frac{b-a}{8}\left\|f^{(2)}\right\|_1$$

and for $k=2$, $(n=4)$

$$|T(a,b,4)| \leq \frac{(b-a)^3}{384}\left\|f^{(4)}\right\|_1.$$

(ii) When n is odd, let $n=2k+1$, $k\geq 0$, and denote

$$Z := \sup_{t\in[a,b]}\left|(t-a)(b-t)^{2k}-(t-a)^{2k}(b-t)\right|.$$

With $P(t,k)$ defined above in Lemma 2, we have

$$P'(t,k) = (t-a)^{2k}+(b-t)^{2k}-2k\left[(t-a)(b-t)^{2k-1}+(t-a)^{2k-1}(b-t)\right],$$

$$P(a,k) = P(b,k) = P\left(\frac{a+b}{2},k\right) = 0,$$

$$P'(a,k) = P'(b,k) = (b-a)^{2k} > 0,$$

$$P'\left(\frac{a+b}{2},k\right) = 2(1-2k)\left(\frac{b-a}{2}\right)^{2k} < 0, \quad \text{for} \quad k\geq 1,$$

so there exists at least one point $\tau\in\left(a,\frac{a+b}{2}\right)$ such that

$$P(\tau,k) := (\tau-a)(b-\tau)^{2k}-(\tau-a)^{2k}(b-\tau) > 0.$$

One can realise that the local extrema for $P(\cdot,k)$ are the real numbers $\tau\in(a,\frac{a+b}{2}]$ that are solutions of the polynomial equation

$$(t-a)^{2k}+(b-t)^{2k}-2k\left[(t-a)(b-t)^{2k-1}+(t-a)^{2k-1}(b-t)\right]=0.$$

Now, by Lemma 2,

$$Z = \max_{\tau}\left\{(\tau-a)(b-\tau)^{2k-1}-(\tau-a)^{2k-1}(b-\tau)\right\} > 0$$

hence

$$|T(a,b,2k)| \leq \max_{\tau}\left\{\frac{(\tau-a)(b-\tau)^{2k-1}-(\tau-a)^{2k-1}(b-\tau)}{2(2k+1)!(b-a)}\right\}\left\|f^{(2k+1)}\right\|_1.$$

For the trivial case $k=0$ $(n=1)$, we have

$$|T(a,b,1)| \leq \frac{1}{2}\|f'\|_1,$$

hence Theorem 5 is proved. ∎

3. Some Examples

In this section we give some examples that highlight Theorems 4 and 5.

(i) For $n=4$, let

$$T(a,b,4) := \frac{f(a)+f(b)}{2}+\frac{b-a}{8}\left[f'(a)-f'(b)\right] + \frac{(b-a)^2}{48}\left[f''(a)+f''(b)\right]-\frac{1}{b-a}\int_a^b f(y)\,dy$$

then

$$|T(a,b,4)| \leq \begin{cases} \frac{(b-a)^{3+\frac{1}{q}}}{24} B^{\frac{1}{q}}(q+1,3q+1)\left\|f^{(4)}\right\|_p, & \text{for } f \in L_p[a,b], \\ & p,q>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \frac{(b-a)^{\frac{7}{2}}}{144\sqrt{10}}\left\|f^{(4)}\right\|_2, & \text{for } f \in L_2[a,b]; \\ \frac{(b-a)^4}{480}\left\|f^{(4)}\right\|_\infty, & \text{for } f \in L_\infty[a,b]; \\ \frac{(b-a)^3}{384}\left\|f^{(4)}\right\|_1, & \text{for } f \in L_1[a,b]. \end{cases}$$

(ii) For $n=5$, let

$$T(a,b,5) := \frac{f(a)+f(b)}{2} + \frac{3(b-a)(f'(a)-f'(b))}{20} + \frac{(b-a)^2(f''(a)+f''(b))}{30} + \frac{(b-a)^3(f'''(a)-f'''(b))}{240} - \frac{1}{b-a}\int_a^b f(y)\,dy,$$

then

$$|T(a,b,5)| \leq \begin{cases} \frac{(b-a)^{4+\frac{1}{q}}}{120} B^{\frac{1}{q}}(q+1,4q+1)\left\|f^{(5)}\right\|_p, & \text{for } f \in L_p[a,b], \\ & p,q>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \frac{(b-a)^{\frac{9}{2}}}{720}\left(\frac{23}{770}\right)^{\frac{1}{2}}\left\|f^{(5)}\right\|_2, & \text{for } f \in L_2[a,b]; \\ \frac{(b-a)^5}{3600}\left\|f^{(5)}\right\|_\infty, & \text{for } f \in L_\infty[a,b]; \\ \frac{(9-\sqrt{6})\sqrt{10\sqrt{6}-15}(b-a)^4}{60000}\left\|f^{(5)}\right\|_1, & \text{for } f \in L_1[a,b]; \end{cases}$$

For $n=5$, $k=2$, solving

$$(\tau-a)^4 + (\tau-b)^4 + 4\left[(\tau-a)(\tau-b)^3 + (\tau-a)^3(\tau-b)\right] = 0$$

we obtain

$$\tau = \frac{a+b}{2} - \frac{\sqrt{10\sqrt{6}-15}}{10}(b-a), \quad \tau \in \left[a, \frac{a+b}{2}\right],$$

for which

$$P(\tau,2) = \frac{(9-\sqrt{6})\sqrt{10\sqrt{6}-15}\,(b-a)^5}{250}$$

is the required maximum in Theorem 5, (ii). Therefore

$$|T(a,b,5)| \leq \frac{(9-\sqrt{6})\sqrt{10\sqrt{6}-15}\,(b-a)^4}{60000}\left\|f^{(5)}\right\|_1.$$

(iii) For $n=6$, let

$$T(a,b,6):=\frac{f(a)+f(b)}{2}+\frac{(b-a)}{6}\left(f'(a)-f'(b)\right)+\frac{(b-a)^2}{24}\left(f''(a)+f''(b)\right)+\frac{(b-a)^3}{144}\left(f'''(a)-f'''(b)\right)+\frac{(b-a)^4}{1440}\left(f^{(4)}(a)+f^{(4)}(b)\right)-\frac{1}{b-a}\int_a^b f(y)\,dy,$$

then

$$|T(a,b,6)|\leq\begin{cases}\frac{(b-a)^{5+\frac{1}{q}}}{720}B^{\frac{1}{q}}(q+1,5q+1)\left\|f^{(6)}\right\|_p, & \text{for } f\in L_p[a,b],\\ & p,q>1,\ \frac{1}{p}+\frac{1}{q}=1;\\ \frac{(b-a)^{\frac{11}{2}}}{1440}\left(\frac{5}{2002}\right)^{\frac{1}{2}}\left\|f^{(6)}\right\|_2, & \text{for } f\in L_2[a,b];\\ \frac{(b-a)^6}{30240}\left\|f^{(6)}\right\|_\infty, & \text{for } f\in L_\infty[a,b];\\ \frac{\left(5\sqrt{10}-14\right)(b-a)^5}{38880}\left\|f^{(6)}\right\|_1, & \text{for } f\in L_1[a,b].\end{cases}$$

For $n=6,\ k=3$,

$$t^*=\frac{a+b}{2}-\frac{\sqrt{6\sqrt{10}-15}\,(b-a)}{6}\in\left[a,\frac{a+b}{2}\right]$$

satisfies

$$(b-t^*)^5-(t^*-a)^5+5\left[(t^*-a)^4(b-t^*)-(t^*-a)(b-t^*)^4\right]=0$$

and

$$M(t^*,3)=\frac{\left(5\sqrt{10}-14\right)(b-a)^6}{27},$$

which provides the required maximum in Theorem 5, (i). Therefore

$$|T(a,b,6)|\leq\frac{\left(5\sqrt{10}-14\right)(b-a)^5}{38880}\left\|f^{(6)}\right\|_1.$$

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