REVERSES OF THE SCHWARZ INEQUALITY IN INNER PRODUCT SPACES GENERALISING A KLAMKIN-MCLENAGHAN RESULT

SEVER S. DRAGOMIR

Abstract. New reverses of the Schwarz inequality in inner product spaces that incorporate the classical Klamkin-McLenaghan result for the case of positive $n-$tuples are given. Applications for Lebesgue integrals are also provided.

1. Introduction

In 2004, the author [1] (see also [3]) proved the following reverse of the Schwarz inequality:

Theorem 1. *Let* $(H;\langle\cdot,\cdot\rangle)$ *be an inner product space over the real or complex number field* $\mathbb{K}$ *and* $x,a\in H$, $r>0$ *such that*

$$\|x-a\|\leq r<\|a\|. \tag{1.1}$$

Then

$$\|x\|\left(\|a\|^2-r^2\right)^{\frac{1}{2}}\leq \operatorname{Re}\langle x,a\rangle \tag{1.2}$$

or, equivalently,

$$\|x\|^2\|a\|^2-[\operatorname{Re}\langle x,a\rangle]^2\leq r^2\|x\|^2. \tag{1.3}$$

The case of equality holds in (1.2) or (1.3) if and only if

$$\|x-a\|=r \quad \text{and} \quad \|x\|^2+r^2=\|a\|^2. \tag{1.4}$$

If above one chooses

$$a=\frac{\Gamma+\gamma}{2}\cdot y \quad \text{and} \quad r=\frac{1}{2}|\Gamma-\gamma|\,\|y\|$$

then the condition (1.1) is equivalent to

$$\left\|x-\frac{\Gamma+\gamma}{2}\cdot y\right\|\leq\frac{1}{2}|\Gamma-\gamma|\,\|y\| \quad \text{and} \quad \operatorname{Re}(\Gamma\bar{\gamma})>0. \tag{1.5}$$

Therefore, we can state the following particular result as well:

Corollary 1. *Let* $(H;\langle\cdot,\cdot\rangle)$ *be as above,* $x,y\in H$ *and* $\gamma,\Gamma\in\mathbb{K}$ *with* $\operatorname{Re}(\Gamma\bar{\gamma})>0$. *If*

$$\left\|x-\frac{\Gamma+\gamma}{2}\cdot y\right\|\leq\frac{1}{2}|\Gamma-\gamma|\,\|y\| \tag{1.6}$$

Date: July 27, 2005.

2000 *Mathematics Subject Classification.* Primary 46C05, 26D15.

Key words and phrases. Schwarz inequality, Reverse inequalities, Inner products, Lebesgue integral.

or, equivalently,

$$\operatorname{Re}\langle \Gamma y - x, x - \gamma y\rangle \geq 0, \tag{1.7}$$

then

$$\begin{aligned} \|x\|\,\|y\| &\leq \frac{\operatorname{Re}\left[\left(\bar{\Gamma}+\bar{\gamma}\right)\langle x,y\rangle\right]}{2\sqrt{\operatorname{Re}(\Gamma\bar{\gamma})}} \\ &= \frac{\operatorname{Re}(\Gamma+\gamma)\operatorname{Re}\langle x,y\rangle + \operatorname{Im}(\Gamma+\gamma)\operatorname{Im}\langle x,y\rangle}{2\sqrt{\operatorname{Re}(\Gamma\bar{\gamma})}} \\ &\left(\leq \frac{|\Gamma+\gamma|}{\sqrt{\operatorname{Re}(\Gamma\bar{\gamma})}}\,|\langle x,y\rangle|\right). \end{aligned} \tag{1.8}$$

The case of equality holds in (1.8) if and only if the equality case holds in (1.6) (or (1.7)) and

$$\|x\| = \sqrt{\operatorname{Re}(\Gamma\bar{\gamma})}\,\|y\|. \tag{1.9}$$

If the restriction $\|a\| > r$ is removed from Theorem 1, then a different reverse of the Schwarz inequality may be stated [2] (see also [3]):

Theorem 2. *Let $(H;\langle\cdot,\cdot\rangle)$ be an inner product space over $\mathbb{K}$ and $x, a \in H$, $r > 0$ such that*

$$\|x - a\| \leq r. \tag{1.10}$$

Then

$$\|x\|\,\|a\| - \operatorname{Re}\langle x,a\rangle \leq \frac{1}{2}r^2. \tag{1.11}$$

The equality holds in (1.11) if and only if the equality case is realised in (1.10) and $\|x\| = \|a\|$.

As a corollary of the above, we can state:

Corollary 2. *Let $(H;\langle\cdot,\cdot\rangle)$ be as above, $x, y \in H$ and $\gamma, \Gamma \in \mathbb{K}$ with $\Gamma \neq -\gamma$. If either (1.6) or, equivalently, (1.7) hold true, then*

$$\|x\|\,\|y\| - \frac{\operatorname{Re}(\Gamma+\gamma)\operatorname{Re}\langle x,y\rangle + \operatorname{Im}(\Gamma+\gamma)\operatorname{Im}\langle x,y\rangle}{|\Gamma+\gamma|} \leq \frac{1}{4}\cdot\frac{|\Gamma-\gamma|^2}{|\Gamma+\gamma|}\|y\|^2. \tag{1.12}$$

The equality holds in (1.12) if and only if the equality case is realised in either (1.6) or (1.7) and

$$\|x\| = \frac{1}{2}|\Gamma+\gamma|\,\|y\|. \tag{1.13}$$

As pointed out in [4], the above results are motivated by the fact that they generalise to the case of real or complex inner product spaces some classical reverses of the Cauchy-Bunyakovsky-Schwarz inequality for positive $n-$tuples due to Polya-Szegö [8], Cassels [10], Shisha-Mond [9] and Greub-Rheinboldt [6].

The main aim of this paper is to establish a new reverse of Schwarz's inequality similar to the ones in Theorems 1 and 2 which will reduce, for the particular case of positive $n-$tuples, to the Klamkin-McLenaghan result from [7].

2. The Results

The following result may be stated.

Theorem 3. *Let* $(H;\langle\cdot,\cdot\rangle)$ *be an inner product space over the real or complex number field* $\mathbb{K}$ *and* $x,a\in H$, $r>0$ *with* $\langle x,a\rangle\neq 0$ *and*

$$\|x-a\|\leq r<\|a\|. \tag{2.1}$$

Then

$$\frac{\|x\|^2}{|\langle x,a\rangle|}-\frac{|\langle x,a\rangle|}{\|a\|^2}\leq\frac{2r^2}{\|a\|\left(\|a\|+\sqrt{\|a\|^2-r^2}\right)}, \tag{2.2}$$

with equality if and only if the equality case holds in (2.1) and

$$\operatorname{Re}\langle x,a\rangle=|\langle x,a\rangle|=\|a\|\left(\|a\|^2-r^2\right)^{\frac{1}{2}}. \tag{2.3}$$

The constant 2 *is best possible in (2.2) in the sese that it cannot be replaced by a smaller quantity.*

Proof. The first condition in (2.1) is obviously equivalent with

$$\frac{\|x\|^2}{|\langle x,a\rangle|}\leq\frac{2\operatorname{Re}\langle x,a\rangle}{|\langle x,a\rangle|}-\frac{\|a\|^2-r^2}{|\langle x,a\rangle|} \tag{2.4}$$

with equality if and only if $\|x-a\|=r$.

Subtracting from both sides of (2.4) the same quantity $\frac{|\langle x,a\rangle|}{\|a\|^2}$ and performing some elementary calculations, we get the equivalent inequality:

$$\frac{\|x\|^2}{|\langle x,a\rangle|}-\frac{|\langle x,a\rangle|}{\|a\|^2} \tag{2.5}$$

$$\leq 2\cdot\frac{\operatorname{Re}\langle x,a\rangle}{|\langle x,a\rangle|}-\left(\frac{|\langle x,a\rangle|^{\frac{1}{2}}}{\|a\|}-\frac{\left(\|a\|^2-r^2\right)^{\frac{1}{2}}}{|\langle x,a\rangle|^{\frac{1}{2}}}\right)^2-\frac{2\sqrt{\|a\|^2-r^2}}{\|a\|}.$$

Since, obviously

$$\operatorname{Re}\langle x,a\rangle\leq|\langle x,a\rangle|\quad\text{and}\quad\left(\frac{|\langle x,a\rangle|^{\frac{1}{2}}}{\|a\|}-\frac{\left(\|a\|^2-r^2\right)^{\frac{1}{2}}}{|\langle x,a\rangle|^{\frac{1}{2}}}\right)^2\geq 0,$$

hence, by (2.5) we get

$$\frac{\|x\|^2}{|\langle x,a\rangle|}-\frac{|\langle x,a\rangle|}{\|a\|^2}\leq 2\left(1-\frac{\sqrt{\|a\|^2-r^2}}{\|a\|}\right) \tag{2.6}$$

with equality if and only if

$$\|x-a\|=r,\quad\operatorname{Re}\langle x,a\rangle=|\langle x,a\rangle|\quad\text{and}\quad|\langle x,a\rangle|=\|a\|\left(\|a\|^2-r^2\right)^{\frac{1}{2}}. \tag{2.7}$$

Observe that (2.6) is equivalent with (2.2) and the first part of the theorem is proved.

To prove the sharpness of the constant, let us assume that there is a $C > 0$ such that

$$\frac{\|x\|^2}{|\langle x,a\rangle|} - \frac{|\langle x,a\rangle|}{\|a\|^2} \le \frac{Cr^2}{\|a\|\left(\|a\| + \sqrt{\|a\|^2 - r^2}\right)}, \tag{2.8}$$

provided $\|x-a\| \le r < \|a\|$.

Now, consider $\varepsilon \in (0,1)$ and let $r = \sqrt{\varepsilon}$, $a, e \in H$, $\|a\| = \|e\| = 1$ and $a \perp e$. Define $x := a + \sqrt{\varepsilon}e$. We observe that $\|x - a\| = \sqrt{\varepsilon} = r < 1 = \|a\|$, which shows that the condition (2.1) of the theorem is fulfilled. We also observe that

$$\|x\|^2 = \|a\|^2 + \varepsilon \|e\|^2 = 1 + \varepsilon, \quad \langle x, a\rangle = \|e\|^2 = 1$$

and utilising (2.8) we get

$$1 + \varepsilon - 1 \le \frac{C\varepsilon}{\left(1 + \sqrt{1-\varepsilon}\right)},$$

giving $1 + \sqrt{1-\varepsilon} \le C$ for any $\varepsilon \in (0,1)$. Letting $\varepsilon \to 0+$, we get $C \ge 2$, which shows that the constant 2 in (2.2) is best possible. ∎

Remark 1. *In a similar manner, one can prove that if* $\operatorname{Re}\langle x,a\rangle \ne 0$ *and (2.2) holds true, then:*

$$\frac{\|x\|^2}{|\operatorname{Re}\langle x,a\rangle|} - \frac{|\operatorname{Re}\langle x,a\rangle|}{\|a\|^2} \le \frac{2r^2}{\|a\|\left(\|a\| + \sqrt{\|a\|^2 - r^2}\right)} \tag{2.9}$$

with equality if and only if $\|x-a\| = r$ *and*

$$\operatorname{Re}\langle x,a\rangle = \|a\|\left(\|a\|^2 - r^2\right)^{\frac{1}{2}}. \tag{2.10}$$

The constant 2 *is best possible in (2.9).*

Remark 2. *Since (2.2) is equivalent with*

$$\|x\|^2\|a\|^2 - |\langle x,a\rangle|^2 \le \frac{2r^2\|a\|^2}{\|a\|\left(\|a\| + \sqrt{\|a\|^2 - r^2}\right)}|\langle x,a\rangle| \tag{2.11}$$

and (2.9) is equivalent to

$$\|x\|^2\|a\|^2 - [\operatorname{Re}\langle x,a\rangle]^2 \le \frac{2r^2\|a\|^2}{\|a\|\left(\|a\| + \sqrt{\|a\|^2 - r^2}\right)}|\operatorname{Re}\langle x,a\rangle| \tag{2.12}$$

hence (2.12) is a tighter inequality than (2.11), because in complex spaces, in general $|\langle x,a\rangle| > |\operatorname{Re}\langle x,a\rangle|$.

The following corollary is of interest.

Corollary 3. *Let* $(H;\langle\cdot,\cdot\rangle)$ *be a real or complex inner product space and* $x, y \in H$ *with* $\langle x,y\rangle \ne 0$, $\gamma, \Gamma \in \mathbb{K}$ *with* $\operatorname{Re}(\Gamma\bar{\gamma}) > 0$. *If either (2.6) or, equivalently (2.7) holds true, then*

$$\frac{\|x\|^2}{|\langle x,y\rangle|} - \frac{|\langle x,y\rangle|}{\|y\|^2} \le |\Gamma + \gamma| - 2\sqrt{\operatorname{Re}(\Gamma\bar{\gamma})}. \tag{2.13}$$

The equality holds in (2.13) if and only if the equality case holds in (2.6) (or in (2.7)) and

$$\operatorname{Re}\left[(\Gamma+\gamma)\langle x,y\rangle\right]=|\Gamma+\gamma|\,|\langle x,y\rangle|=|\Gamma+\gamma|\sqrt{\operatorname{Re}\left(\Gamma\bar{\gamma}\right)}\,\|y\|^{2}. \tag{2.14}$$

Proof. We use the inequality (2.2) in its equivalent form

$$\frac{\|x\|^{2}}{|\langle x,a\rangle|}-\frac{|\langle x,a\rangle|}{\|a\|^{2}}\leq\frac{2\left(\|a\|-\sqrt{\|a\|^{2}-r^{2}}\right)}{\|a\|}.$$

Choosing $a=\frac{\Gamma+\gamma}{2}\cdot y$ and $r=\frac{1}{2}|\Gamma-\gamma|\,\|y\|$, we have

$$\frac{\|x\|^{2}}{\left|\frac{\Gamma+\gamma}{2}\right||\langle x,y\rangle|}-\frac{\left|\frac{\Gamma+\gamma}{2}\right||\langle x,y\rangle|}{\left|\frac{\Gamma+\gamma}{2}\right|^{2}\|y\|^{2}}\leq\frac{2\left(\left|\frac{\Gamma+\gamma}{2}\right|\|y\|-\sqrt{\left|\frac{\Gamma+\gamma}{2}\right|^{2}\|y\|^{2}-\frac{1}{4}|\Gamma-\gamma|^{2}\|y\|^{2}}\right)}{\left|\frac{\Gamma-\gamma}{2}\right|\|y\|}$$

which is equivalent to (2.13). ∎

Remark 3. *The inequality (2.13) has been obtained in a different way in* [5, Theorem 2]. *However, in* [5] *the authors did not consider the equality case which may be of interest for applications.*

Remark 4. *If we assume that* $\Gamma=M\geq m=\gamma>0$, *which is very convenient in applications, then*

$$\frac{\|x\|^{2}}{|\langle x,y\rangle|}-\frac{|\langle x,y\rangle|}{\|y\|^{2}}\leq\left(\sqrt{M}-\sqrt{m}\right)^{2}, \tag{2.15}$$

provided that either

$$\operatorname{Re}\langle My-x,x-my\rangle\geq 0 \tag{2.16}$$

or, equivalently,

$$\left\|x-\frac{m+M}{2}y\right\|\leq\frac{1}{2}(M-m)\|y\| \tag{2.17}$$

holds true.

The equality holds in (2.15) if and only if the equality case holds in (2.16) (or in (2.17)) and

$$\operatorname{Re}\langle x,y\rangle=|\langle x,y\rangle|=\sqrt{Mm}\,\|y\|^{2}. \tag{2.18}$$

The multiplicative constant $C=1$ in front of $\left(\sqrt{M}-\sqrt{m}\right)^{2}$ cannot be replaced in general with a smaller positive quantity.

Now for a non-zero complex number z, we define $\operatorname{sgn}(z):=\frac{z}{|z|}$.

The following result may be stated:

Proposition 1. *Let $(H;\langle\cdot,\cdot\rangle)$ be a real or complex inner product space and $x,y\in H$ with $\operatorname{Re}\langle x,y\rangle\neq 0$ and $\gamma,\Gamma\in\mathbb{K}$ with $\operatorname{Re}(\Gamma\bar{\gamma})>0$. If either (2.6) or, equivalently, (2.7) hold true, then*

$$
\begin{aligned}
(2.19)\qquad & (0\leq \|x\|^2\|y\|^2-|\langle x,y\rangle|^2\leq) \\
& \|x\|^2\|y\|^2-\left[\operatorname{Re}\left(\operatorname{sgn}\left(\frac{\Gamma+\gamma}{2}\right)\cdot\langle x,y\rangle\right)\right]^2 \\
& \leq\left(|\Gamma+\gamma|-2\sqrt{\operatorname{Re}(\Gamma\bar{\gamma})}\right)\left|\operatorname{Re}\left(\operatorname{sgn}\left(\frac{\Gamma+\gamma}{2}\right)\cdot\langle x,y\rangle\right)\right|\|y\|^2 \\
& \left(\leq\left(|\Gamma+\gamma|-2\sqrt{\operatorname{Re}(\Gamma\bar{\gamma})}\right)|\langle x,y\rangle|\,\|y\|^2\right).
\end{aligned}
$$

The equality holds in (2.19) if and only if the equality case holds in (2.6) (or in (2.7)) and

$$\operatorname{Re}\left[\operatorname{sgn}\left(\frac{\Gamma+\gamma}{2}\right)\cdot\langle x,y\rangle\right]=\sqrt{\operatorname{Re}(\Gamma\bar{\gamma})}\,\|y\|^2.$$

Proof. The inequality (2.9) is equivalent with:

$$\|x\|^2\|a\|^2-[\operatorname{Re}\langle x,a\rangle]^2\leq 2\left(\|a\|+\sqrt{\|a\|^2-r^2}\right)\cdot|\operatorname{Re}\langle x,a\rangle|\,\|a\|.$$

If in this inequality we choose $a=\frac{\Gamma+\gamma}{2}\cdot y$ and $r=\frac{1}{2}|\Gamma-\gamma|\,|y$, we have

$$
\begin{aligned}
\|x\|^2\left|\frac{\Gamma+\gamma}{2}\right|^2\|y\|^2-\left(\operatorname{Re}\left[\left(\frac{\Gamma+\gamma}{2}\right)\cdot\langle x,y\rangle\right]\right)^2 \\
\leq 2\left(\left|\frac{\Gamma+\gamma}{2}\right|\|y\|-\sqrt{\left|\frac{\Gamma+\gamma}{2}\right|^2\|y\|^2-\frac{1}{4}|\Gamma-\gamma|^2\|y\|^2}\right) \\
\times\left|\operatorname{Re}\left[\left(\frac{\Gamma+\gamma}{2}\right)\cdot\langle x,y\rangle\right]\right|\left|\frac{\Gamma+\gamma}{2}\right|\|y\|,
\end{aligned}
$$

which, on dividing by $\left|\frac{\Gamma+\gamma}{2}\right|^2\neq 0$ (since $\operatorname{Re}(\Gamma\bar{\gamma})>0$), is clearly equivalent to (2.19). ∎

Remark 5. *If we assume that x,y,m,M satisfy either (2.16) or, equivalently (2.17), then*

$$\frac{\|x\|^2}{|\operatorname{Re}\langle x,y\rangle|}-\frac{|\operatorname{Re}\langle x,y\rangle|}{\|y\|^2}\leq\left(\sqrt{M}-\sqrt{m}\right)^2 \tag{2.20}$$

or, equivalently

$$\|x\|^2\|y\|^2-[\operatorname{Re}\langle x,y\rangle]^2\leq\left(\sqrt{M}-\sqrt{m}\right)^2|\operatorname{Re}\langle x,y\rangle|\,\|y\|^2. \tag{2.21}$$

The equality holds in (2.20) (or (2.21)) if and only if the case of equality is valid in (2.16) (or (2.17)) and

$$\operatorname{Re}\langle x,y\rangle=\sqrt{Mm}\,\|y\|^2. \tag{2.22}$$

3. Applications for Integrals

Let (Ω, Σ, μ) be a measure space consisting of a set Ω, a $\sigma-$algebra of parts Σ and a countably additive and positive measure μ on Σ with values in $\mathbb{R} \cup \{\infty\}$.

Denote by $L^2_\rho(\Omega, \mathbb{K})$ the Hilbert space of all $\mathbb{K}$–valued functions f defined on Ω that are $2-\rho-$integrable on Ω, i.e., $\int_\Omega \rho(t) |f(s)|^2 d\mu(s) < \infty$, where $\rho : \Omega \to [0, \infty)$ is a measurable function on Ω.

The following proposition contains a reverse of the Cauchy-Bunyakovsky-Schwarz integral inequality:

Proposition 2. *Let $f, g \in L^2_\rho(\Omega, \mathbb{K})$, $r > 0$ be such that*

$$\int_\Omega \rho(t) |f(t) - g(t)|^2 d\mu(t) \le r^2 < \int_\Omega \rho(t) |g(t)|^2 d\mu(t). \tag{3.1}$$

Then

$$\begin{aligned} &\int_\Omega \rho(t) |f(t)|^2 d\mu(t) \int_\Omega \rho(t) |g(t)|^2 d\mu(t) - \left| \int_\Omega \rho(t) f(t) \overline{g(t)} d\mu(t) \right|^2 \\ &\quad \le 2 \left(\int_\Omega \rho(t) |g(t)|^2 d\mu(t) \right)^{\frac{1}{2}} \left| \int_\Omega \rho(t) f(t) \overline{g(t)} d\mu(t) \right| \\ &\quad\quad \times \left[\left(\int_\Omega \rho(t) |g(t)|^2 d\mu(t) \right)^{\frac{1}{2}} - \left(\int_\Omega \rho(t) |g(t)|^2 d\mu(t) - r^2 \right)^{\frac{1}{2}} \right]. \end{aligned} \tag{3.2}$$

The constant 2 is sharp in (3.2).

The proof follows from Theorem 3 applied for the Hilbert space $\left(L^2_\rho(\Omega, \mathbb{K}), \langle\cdot,\cdot\rangle_\rho\right)$ where

$$\langle f, g \rangle_\rho := \int_\Omega \rho(t) f(t) \overline{g(t)} d\mu(t).$$

Remark 6. *We observe that if $\int_\Omega \rho(t) d\mu(t) = 1$, then a simple sufficient condition for (3.1) to hold is*

$$|f(t) - g(t)| \le r < |g(t)| \quad \text{for} \quad \mu - a.e. \quad t \in \Omega. \tag{3.3}$$

The second general integral inequality is incorporated in:

Proposition 3. *Let $f, g \in L^2_\rho(\Omega, \mathbb{K})$ and $\Gamma, \gamma \in \mathbb{K}$ with $\operatorname{Re}(\Gamma\bar{\gamma}) > 0$. If either*

$$\int_\Omega \operatorname{Re}\left[(\Gamma g(t) - f(t)) \left(\overline{f(t)} - \bar{\gamma} \overline{g(t)} \right) \right] \rho(t) d\mu(t) \ge 0 \tag{3.4}$$

or, equivalently,

$$\begin{aligned} &\left(\int_\Omega \rho(t) \left| f(t) - \frac{\Gamma + \gamma}{2} g(t) \right|^2 d\mu(t) \right)^{\frac{1}{2}} \\ &\qquad\qquad \le \frac{1}{2} |\Gamma - \gamma| \left(\int_\Omega \rho(t) |g(t)|^2 d\mu(t) \right)^{\frac{1}{2}} \end{aligned} \tag{3.5}$$

holds, then

$$\int_{\Omega}\rho(t)|f(t)|^2 d\mu(t)\int_{\Omega}\rho(t)|g(t)|^2 d\mu(t)-\left|\int_{\Omega}\rho(t)f(t)\overline{g(t)}d\mu(t)\right|^2 \\ \leq\left[|\Gamma+\gamma|-2\sqrt{\operatorname{Re}(\Gamma\bar{\gamma})}\right]\left|\int_{\Omega}\rho(t)f(t)\overline{g(t)}d\mu(t)\right|\int_{\Omega}\rho(t)|g(t)|^2 d\mu(t). \tag{3.6}$$

The proof is obvious by Corollary 3.

Remark 7. *A simple sufficient condition for the inequality (3.4) to hold is:*

$$\operatorname{Re}\left[(\Gamma g(t)-f(t))\left(\overline{f(t)}-\bar{\gamma}\overline{g(t)}\right)\right]\geq 0, \tag{3.7}$$

for $\mu-$a.e. $t\in\Omega$.

A more convenient result that may be useful in applications is:

Corollary 4. *If $f,g\in L^2_{\rho}(\Omega,\mathbb{K})$ and $M\geq m>0$ such that either*

$$\int_{\Omega}\operatorname{Re}\left[(Mg(t)-f(t))\left(\overline{f(t)}-m\overline{g(t)}\right)\right]f(t)\,d\mu(t)\geq 0 \tag{3.8}$$

or, equivalently,

$$\left(\int_{\Omega}\rho(t)\left|f(t)-\frac{M+m}{2}g(t)\right|^2 d\mu(t)\right)^{\frac{1}{2}} \\ \leq\frac{1}{2}(M-m)\left(\int_{\Omega}\rho(t)|g(t)|^2 d\mu(t)\right)^{\frac{1}{2}}, \tag{3.9}$$

holds, then

$$\int_{\Omega}\rho(t)|f(t)|^2 d\mu(t)\int_{\Omega}\rho(t)|g(t)|^2 d\mu(t)-\left|\int_{\Omega}\rho(t)f(t)\overline{g(t)}d\mu(t)\right|^2 \\ \leq\left(\sqrt{M}-\sqrt{m}\right)^2\left|\int_{\Omega}\rho(t)f(t)\overline{g(t)}d\mu(t)\right|\int_{\Omega}\rho(t)|g(t)|^2 d\mu(t). \tag{3.10}$$

Remark 8. *Since, obviously,*

$$\operatorname{Re}\left[(Mg(t)-f(t))\left(\overline{f(t)}-m\overline{g(t)}\right)\right] \\ =(M\operatorname{Re}g(t)-\operatorname{Re}f(t))(\operatorname{Re}f(t)-m\operatorname{Re}g(t)) \\ +(M\operatorname{Im}g(t)-\operatorname{Im}f(t))(\operatorname{Im}f(t)-m\operatorname{Im}g(t))$$

for any $t\in\Omega$, hence a very simple sufficient condition that can be useful in practical applications for (3.8) to hold is:

$$M\operatorname{Re}g(t)\geq\operatorname{Re}f(t)\geq m\operatorname{Re}g(t)$$

and

$$M\operatorname{Im}g(t)\geq\operatorname{Im}f(t)\geq m\operatorname{Im}g(t)$$

for μ-a.e. $t\in\Omega$.

If the functions are in $L_{\rho}^{2}(\Omega,\mathbb{R})$ (here $\mathbb{K}=\mathbb{R}$), and $f,g\geq 0$, $g(t)\neq 0$ for μ-a.e. $t\in\Omega$, then one can state the result:

$$\int_{\Omega}\rho(t)\,f^{2}(t)\,d\mu(t)\int_{\Omega}\rho(t)\,g^{2}(t)\,d\mu(t)-\left(\int_{\Omega}\rho(t)\,f(t)\,g(t)\,d\mu(t)\right)^{2} \leq\left(\sqrt{M}-\sqrt{m}\right)^{2}\int_{\Omega}\rho(t)\,f(t)\,g(t)\,d\mu(t)\int_{\Omega}\rho(t)\,g^{2}(t)\,d\mu(t), \tag{3.11}$$

provided

$$0\leq m\leq\frac{f(t)}{g(t)}\leq M<\infty\quad\text{for}\quad\mu-\text{a.e.}\quad t\in\Omega. \tag{3.12}$$

Remark 9. *We notice that (3.11) is a generalisation for the abstract Lebesgue integral of the Klamkin-McLenaghan inequality* [7]

$$\frac{\sum_{k=1}^{n}w_{k}x_{k}^{2}}{\sum_{k=1}^{n}w_{k}x_{k}y_{k}}-\frac{\sum_{k=1}^{n}w_{k}x_{k}y_{k}}{\sum_{k=1}^{n}w_{k}y_{k}^{2}}\leq\left(\sqrt{M}-\sqrt{m}\right)^{2}, \tag{3.13}$$

provided the nonnegative real numbers x_{k},y_{k} $(k\in\{1,\dots,n\})$ *satisfy the assumption*

$$0\leq m\leq\frac{x_{k}}{y_{k}}\leq M<\infty\quad\textit{for each}\quad k\in\{1,\dots,n\} \tag{3.14}$$

and $w_{k}\geq 0,\ k\in\{1,\dots,n\}$.

We also remark that Klamkin-McLenaghan inequality (3.13) is a generalisation in its turn of the Shisha-Mond inequality obtained earlier in [9]*:*

$$\frac{\sum_{k=1}^{n}a_{k}^{2}}{\sum_{k=1}^{n}a_{k}b_{k}}-\frac{\sum_{k=1}^{n}a_{k}b_{k}}{\sum_{k=1}^{n}b_{k}^{2}}\leq\left(\sqrt{\frac{A}{b}}-\sqrt{\frac{a}{B}}\right)^{2}$$

provided

$$0<a\leq a_{k}\leq A,\quad 0<b\leq b_{k}\leq B$$

for each $k\in\{1,\dots,n\}$.

References

[1] S.S. DRAGOMIR, Reverse of Schwarz, triangle and Bessel inequalities in inner product spaces, *J. Inequal. Pure & Appl. Math.,* **5**(3) (2004), Art. 76. [ONLINE `http://jipam.vu.edu.au/article.php?sid=432`]

[2] S.S. DRAGOMIR, New reverses of Schwarz, triangle and Bessel inequalities in inner product spaces, *Austral. J. Math. Anal. & Applics.,* **1**(1) (2004), Art. 1. [ONLINE : http://ajmaa.org/cgi-bin/paper.pl?string=nrstbiips.tex].

[3] S.S. DRAGOMIR, *Advances in Inequalities of the Schwarz, Grüss and Bessel Type in Inner Product Spaces,* Nova Science Publishers Inc., New York, 2005.

[4] S.S. DRAGOMIR, Reverses of the Cauchy-Bunyakovsky-Schwarz inequality for $n-$tuples of complex numbers, *Bull. Austral. Math. Soc.,* **69** (2004), 465-480.

[5] N. ELEZOVIĆ, L. MARANGUNIĆ and J. PEČARIĆ, Unified treatment of complemented Schwarz and Gruss inequalities in inner product spaces, *Math. Ineq. & Appl.,* **8**(2) (2005), 223-232.

[6] W. GREUB and W. RHEINBOLDT, On a generalisation of an inequality of L.V. Kantorovich, *Proc. Amer. Math. Soc.,* **10** (1959), 407-415.

[7] M.S. KLAMKIN and K.G. McLENAGHAN, An ellipse inequality, *Math. Mag.,* **50** (1977), 261-263.

[8] G. PÓLYA and G. SZEGÖ, *Aufgaben und Lehrsätze aus der Analysis,* Vol. 1, Berlin 1925, pp. 57 and 213-214.

[9] O. SHISHA and B. MOND, Bounds on differences of means, *Inequalities I,* New York-London, 1967, 293-308.

[10] G.S. WATSON, Serial correlation in regression analysis I, *Biometrika*, **42** (1955), 327-342.

School of Computer Science and Mathematics, Victoria University, PO Box 14428, Melbourne City, VIC 8001, Australia.
E-mail address: sever.dragomir@vu.edu.au
URL: http://rgmia.vu.edu.au/dragomir