REFINEMENT OF INEQUALITIES AMONG MEANS

INDER JEET TANEJA

ABSTRACT. In this paper we shall consider some famous *means* such as *arithmetic*, *harmonic*, *geometric*, *root-square means*, etc. Some new *means* recently studied are also presented. Different kinds of refinement of inequalities among these *means* are given.

1. MEAN OF ORDER t

Let us consider the following well known *mean of order* t:

$$B_t(a,b) = \begin{cases} \left(\frac{a^t+b^t}{2}\right)^{1/t}, & t \neq 0 \\ \sqrt{ab}, & t = 0 \\ \max\{a,b\}, & t = \infty \\ \min\{a,b\}, & t = -\infty \end{cases} \tag{1.1}$$

for all $a, b, t \in \mathbb{R},\ a, b > 0$.

In particular, we have

$$B_{-1}(a,b) = H(a,b) = \frac{2ab}{a+b},$$
$$B_0(a,b) = G(a,b) = \sqrt{ab},$$
$$B_{1/2}(a,b) = N_1(a,b) = \left(\frac{\sqrt{a}+\sqrt{b}}{2}\right)^2,$$
$$B_1(a,b) = A(a,b) = \frac{a+b}{2},$$

and

$$B_2(a,b) = S(a,b) = \sqrt{\frac{a^2+b^2}{2}}.$$

The means, $H(a,b)$, $G(a,b)$, $A(a,b)$ and $S(a,b)$ are known in the literature as *harmonic*, *geometric*, *arithmetic* and *root-square means* respectively. For simplicity we can call the measure, $N_1(a,b)$ as *square-root* mean. It is well know that [1] the *mean of order* s given in (1.1) is monotonically increasing in s, then we can write

$$H(a,b) \leqslant G(a,b) \leqslant N_1(a,b) \leqslant A(a,b) \leqslant S(a,b). \tag{1.2}$$

Dragomir and Pearce [3] (page 242) proved the following inequality:

$$\frac{a^r+b^r}{2} \leqslant \frac{b^{r+1}-a^{r+1}}{(r+1)(b-a)} \leqslant \left(\frac{a+b}{2}\right)^r, \tag{1.3}$$

2000 *Mathematics Subject Classification.* 26D15; 26D10.
Key words and phrases. Arithmetic mean; geometric mean; harmonic mean; root-square mean; square-root mean.

for all $a, b > 0$, $a \neq b$, $r \in (0,1)$. In particular take $r = \frac{1}{2}$ in (1.3), we get

$$\frac{\sqrt{a}+\sqrt{b}}{2} \leqslant \frac{2(b^{3/2}-a^{3/2})}{3(b-a)} \leqslant \sqrt{\frac{a+b}{2}},\ a \neq b. \tag{1.4}$$

After necessary calculations in (1.5), we get

$$\left(\frac{\sqrt{a}+\sqrt{b}}{2}\right)^2 \leqslant \frac{a+\sqrt{ab}+b}{3} \leqslant \left(\frac{\sqrt{a}+\sqrt{b}}{2}\right)\left(\sqrt{\frac{a+b}{2}}\right). \tag{1.5}$$

On the other side we can easily check that

$$\left(\frac{\sqrt{a}+\sqrt{b}}{2}\right)\left(\sqrt{\frac{a+b}{2}}\right) \leqslant \frac{a+b}{2}. \tag{1.6}$$

Finally, the expressions (1.2), (1.5) and (1.6) lead us to the following inequality:

$$H(a,b) \leqslant G(a,b) \leqslant N_1(a,b) \leqslant N_3(a,b) \leqslant N_2(a,b) \leqslant A(a,b) \leqslant S(a,b), \tag{1.7}$$

where

$$N_2(a,b) = \left(\frac{\sqrt{a}+\sqrt{b}}{2}\right)\left(\sqrt{\frac{a+b}{2}}\right),$$

and

$$N_3(a,b) = \frac{a+\sqrt{ab}+b}{3}.$$

Moreover, we can write

$$N_1(a,b) = \frac{A(a,b)+G(a,b)}{2},$$

$$N_2(a,b) = \sqrt{N_1(a,b)A(a,b)},$$

and

$$N_3(a,b) = \frac{2A(a,b)+G(a,b)}{3}.$$

Thus we have three new means, where $N_1(a,b)$ appears as a natural way. The $N_2(a,b)$ can be seen in Taneja [4, 5] and the mean $N_3(a,b)$ is known as Heron's mean [2]. Some studies on it can be seen in Zhang and Wu [6].

2. Difference of Means and Their Convexity

Let us consider the following *difference of means*:

$$M_{SA}(a,b) = S(a,b) - A(a,b), \tag{2.1}$$

$$M_{SN_2}(a,b) = S(a,b) - N_2(a,b), \tag{2.2}$$

$$M_{SN_3}(a,b) = S(a,b) - N_3(a,b), \tag{2.3}$$

$$M_{SN_1}(a,b) = S(a,b) - N_1(a,b), \tag{2.4}$$

$$M_{SG}(a,b) = S(a,b) - G(a,b), \tag{2.5}$$

$$M_{SH}(a,b) = S(a,b) - H(a,b), \tag{2.6}$$

$$M_{AN_2}(a,b) = A(a,b) - N_2(a,b), \tag{2.7}$$

$$M_{AG}(a,b) = A(a,b) - G(a,b), \tag{2.8}$$

$$M_{AH}(a,b) = A(a,b) - H(a,b), \tag{2.9}$$

$$M_{N_2N_1}(a,b) = N_2(a,b) - N_1(a,b), \tag{2.10}$$

and

$$M_{N_2G}(a,b) = N_2(a,b) - G(a,b). \tag{2.11}$$

We easily check that

$$\begin{aligned} M_{AG}(a,b) &= 2\left[N_1(a,b) - G(a,b)\right] := 2M_{N_1G}(a,b) \\ &= 2\left[A(a,b) - N_1(a,b)\right] := 2M_{AN_1}(a,b) \\ &= 3\left[A(a,b) - N_3(a,b)\right] := 3M_{AN_3}(a,b) \\ &= \frac{3}{2}\left[N_3(a,b) - G(a,b)\right] := \frac{3}{2}M_{N_3G}(a,b) \\ &= 6\left[N_3(a,b) - N_1(a,b)\right] := 6M_{N_3N_1}(a,b). \end{aligned} \tag{2.12}$$

Now, we shall prove the convexity of the means (2.1)-(2.11). It is based on the following lemma.

Lemma 2.1. *Let $f : I \subset \mathbb{R}_+ \to \mathbb{R}$ be a convex and differentiable function satisfying $f(1) = f'(1) = 0$. Consider a function*

$$\phi_f(a,b) = af\left(\frac{b}{a}\right), \ a,b > 0, \tag{2.13}$$

then the function $\phi_f(a,b)$ is convex in $\mathbb{R}^2_+$, and satisfies the following inequality:

$$0 \leqslant \phi_f(a,b) \leqslant \left(\frac{b-a}{a}\right)\phi'_f(a,b). \tag{2.14}$$

Proof. It is well known that for the convex and differentiable function f, we have the inequality

$$f'(x)(y-x) \leqslant f(y) - f(x) \leqslant f'(y)(y-x), \tag{2.15}$$

for all $x, y \in \mathbb{R}_+$.

Take $y = \frac{b}{a}$ and $x = 1$ in (2.15) one gets

$$f'(1)\left(\frac{b}{a} - 1\right) \leqslant f\left(\frac{b}{a}\right) - f(1) \leqslant f'\left(\frac{b}{a}\right)\left(\frac{b}{a} - 1\right),$$

or equivalently,

$$f'(1)\,(b-a) \leqslant af\left(\frac{b}{a}\right) - af(1) \leqslant af'\left(\frac{b}{a}\right)\left(\frac{b-a}{a}\right) \tag{2.16}$$

Since $f(1) = f'(1) = 0$, then from (2.16) we get (2.14).

Now we shall show that the function $\phi_f(a,b)$ is jointly convex in a and b. Since the function f is convex, then for any (x_1,y_1), $(x_2,y_2) \in \mathbb{R}^2_+$, $0 < \lambda_1, \lambda_2 < 1$, $\lambda_1 + \lambda_2 = 1$ we can write

$$f\left(\frac{\lambda_1 x_1 + \lambda_2 x_2}{\lambda_1 y_1 + \lambda_2 y_2}\right) = f\left(\frac{\lambda_1 y_1 x_1}{y_1(\lambda_1 y_1 + \lambda_2 y_2)} + \frac{\lambda_2 y_2 x_2}{y_2(\lambda_1 y_1 + \lambda_2 y_2)}\right).$$

$$\leqslant \frac{\lambda_1 y_1}{\lambda_1 y_1 + \lambda_2 y_2} f\left(\frac{x_1}{y_1}\right) + \frac{\lambda_2 y_2}{\lambda_1 y_1 + \lambda_2 y_2} f\left(\frac{x_2}{y_2}\right). \tag{2.17}$$

Multiply (2.17) by $\lambda_1 y_1 + \lambda_2 y_2$ one gets

$$(\lambda_1 y_1 + \lambda_2 y_2) f\left(\frac{\lambda_1 x_1 + \lambda_2 x_2}{\lambda_1 y_1 + \lambda_2 y_2}\right) \leqslant \lambda_1 y_1 f\left(\frac{\lambda_1 x_1}{\lambda_1 y_1}\right) + \lambda_2 y_2 f\left(\frac{\lambda_2 x_2}{\lambda_2 y_2}\right),$$

i.e.,

$$\phi_f(\lambda_1 x_1 + \lambda_2 x_2, \lambda_1 y_1 + \lambda_2 y_2) \leqslant \lambda_1 \phi_f(x_1, y_1) + \lambda_2 \phi_f(x_2, y_2), \tag{2.18}$$

for any (x_1,y_1), $(x_2,y_2) \in \mathbb{R}^2_+$. The expression (2.18) completes the required proof. □

Now we shall show that the *difference of means* given by (2.1)-(2.11) are *convex* in $\mathbb{R}^2_+$ Later in Section 3 we shall apply the convexity of these functions to establish improvement over the inequality (1.7).

Theorem 2.1. *The difference of means given by (2.1)-(2.11) are nonnegative and convex in* $\mathbb{R}^2_+$.

Proof. We shall write each measure in the form of generating function according to the measure (2.13), and then give their first and second order derivatives. It is understood that $x \in (0,\infty)$.

- **For** $M_{SA}(a,b)$**:**

$$f_{SA}(x) = \sqrt{\frac{x^2+1}{2}} - \frac{x+1}{2},$$
$$f'_{SA}(x) = \frac{x}{\sqrt{2}\sqrt{x^2+1}} - \frac{1}{2},$$

and

$$f''_{SA}(x) = \frac{2}{(2x^2+2)^{3/2}} > 0.$$

- **For** $M_{SN_3}(a,b)$**:**

$$f_{SN_3}(x) = \sqrt{\frac{x^2+1}{2}} - \frac{x+\sqrt{x}+1}{3},$$
$$f'_{SN_3}(x) = \frac{6x^{3/2} - (2\sqrt{x}+1)\sqrt{2(x^2+1)}}{6\sqrt{2x(x^2+1)}},$$

and

$$f''_{SN_3}(x) = \frac{24x^{3/2} + \left(2x^2+2\right)^{3/2}}{12x^{3/2}(2x^2+2)^{3/2}} > 0.$$

- **For** $M_{SN_2}(a,b)$**:**

$$f_{SN_2}(x) = \frac{2\sqrt{x^2+1} - \left(\sqrt{x}+1\right)\sqrt{x+1}}{2\sqrt{2}},$$

$$f'_{SN_2}(x) = \frac{4x^{3/2}\sqrt{x+1} - \left(2x+\sqrt{x}+1\right)\sqrt{x^2+1}}{4\sqrt{2x(x+1)(x^2+1)}},$$

and

$$f''_{SN_2}(x) = \frac{\left(x^{3/2}+1\right)\left(x^2+1\right)^{3/2} + 8x^{3/2}(x+1)^{3/2}}{8\sqrt{2}\left[x(x+1)(x^2+1)\right]^{3/2}} > 0.$$

- **For** $M_{SN_1}(a,b)$**:**

$$f_{SN_1}(x) = \frac{2\sqrt{2(x^2+1)} - \left(\sqrt{x}+1\right)^2}{4},$$

$$f'_{SN_1}(x) = \frac{4x^{3/2} - \left(\sqrt{x}+1\right)\sqrt{2(x^2+1)}}{4\sqrt{2x(x^2+1)}},$$

and

$$f''_{SN_1}(x) = \frac{16x^{5/2} + x(2x^2+2)^{3/2}}{8x^{5/2}(2x^2+2)^{3/2}} > 0.$$

- **For** $M_{SG}(a,b)$**:**

$$f_{SG}(x) = \sqrt{\frac{x^2+1}{2}} - \sqrt{x},$$

$$f'_{SG}(x) = \frac{\sqrt{2}x^{3/2} - \sqrt{x^2+1}}{2\sqrt{x(x^2+1)}},$$

and

$$f''_{SG}(x) = \frac{1}{\sqrt{2}(x^2+1)^{3/2}} + \frac{1}{4x^{3/2}} > 0.$$

- **For** $M_{SH}(a,b)$**:**

$$f_{SH}(x) = \sqrt{\frac{x^2+1}{2}} - \frac{2x}{x+1},$$

$$f'_{SH}(x) = \frac{x(x+1)^2 - 2\sqrt{2(x^2+1)}}{(x+1)^2\sqrt{2(x^2+1)}},$$

and

$$f''_{SH}(x) = \frac{2\left[(x+1)^3 + 2(2x^2+2)^{3/2}\right]}{(x+1)^3(2x^2+2)^{3/2}} > 0.$$

- **For** $M_{AN_2}(a,b)$**:**

$$f_{AN_2}(x) = \frac{2(x+1) - (\sqrt{x}+1)\sqrt{2(x+1)}}{4},$$
$$f'_{AN_2}(x) = \frac{2\sqrt{2x(x+1)} - (2x+\sqrt{x}+1)}{4\sqrt{2(x+1)}},$$

and

$$f''_{AN_2}(x) = \frac{x^{3/2}+1}{4x^{3/2}(2x+2)^{3/2}} > 0.$$

- **For** $M_{AG}(a,b)$**:**

$$f_{AG}(x) = \frac{1}{2}(\sqrt{x}-1)^2,$$
$$f'_{AG}(x) = \frac{\sqrt{x}-1}{2\sqrt{x}}$$

and

$$f''_{AG}(x) = \frac{1}{4x^{3/2}} > 0.$$

- **For** $M_{AH}(a,b)$**:**

$$f_{AH}(x) = \frac{(x-1)^2}{2(x+1)},$$
$$f'_{AH}(x) = \frac{(x-1)(x+3)}{2(x+1)^2}$$

and

$$f''_{AH}(x) = \frac{4}{(x+1)^3} > 0.$$

- **For** $M_{N_2N_1}(a,b)$**:**

$$f_{N_2N_1}(x) = \frac{(\sqrt{x}+1)\sqrt{2(x+1)} - (\sqrt{x}+1)^2}{4},$$
$$f'_{N_2N_1}(x) = \frac{2x+\sqrt{x}+1 - (\sqrt{x}+1)\sqrt{2(x+1)}}{4\sqrt{2x(x+1)}},$$

and

$$f''_{N_2N_1}(x) = \frac{(2x+2)^{3/2} - 2(x^{3/2}+1)}{8x^{3/2}(2x+2)^{3/2}}.$$

Since $(x+1)^{3/2} \geqslant x^{3/2}+1$, $\forall x \in (0,\infty)$ and $2^{3/2} \geqslant 2$, then obviously, $f''_{N_2N_1}(x) \geqslant 0$, $\forall x \in (0,\infty)$.

• **For** $M_{N_2G}(a,b)$**:**

$$f_{N_2G}(x) = \frac{(\sqrt{x}+1)\sqrt{2(x+1)} - 4x}{4},$$
$$f'_{N_2G}(x) = \frac{2x+1+\sqrt{x} - 2\sqrt{2(x+1)}}{4\sqrt{2x(x+1)}},$$

and

$$f''_{N_2G}(x) = \frac{(2x+2)^{3/2} - (x^{3/2}+1)}{4x^{3/2}(2x+2)^{3/2}}.$$

Since $(x+1)^{3/2} \geqslant x^{3/2}+1$, $\forall x \in (0,\infty)$ and $2^{3/2} \geqslant 1$, then obviously, $f''_{N_2G}(x) \geqslant 0$, $\forall x \in (0,\infty)$.

We see that in all the cases the generating function $f_{(\cdot)}(1) = f'_{(\cdot)}(1) = 0$ and the second derivative is positive for all $x \in (0,\infty)$. This proves the *nonegativity* and *convexity* of the means (2.1)-(2.16) in $\mathbb{R}^2_+$. This completes the proof of the theorem. □

Remark 2.1. *The inequality (1.7) also present more nonnegative differences but we have considered only the convex ones.*

3. Inequality Among Difference of Means

In view of (1.7), the following inequalities are obviously true:

(3.1) $M_{SA}(a,b) \leqslant M_{SN_2}(a,b) \leqslant M_{SN_3}(a,b) \leqslant M_{SN_1}(a,b) \leqslant M_{SG}(a,b) \leqslant M_{SH}(a,b)$,

(3.2) $M_{AN_2}(a,b) \leqslant M_{AN_3}(a,b) \leqslant M_{AN_1}(a,b) \leqslant M_{AG}(a,b) \leqslant M_{AH}(a,b)$,

(3.3) $M_{N_2N_3}(a,b) \leqslant M_{N_2N_1}(a,b) \leqslant M_{N_2G}(a,b) \leqslant M_{N_2H}(a,b)$,

(3.4) $M_{N_3N_1}(a,b) \leqslant M_{N_3G}(a,b) \leqslant M_{N_3H}(a,b)$,

and

(3.5) $M_{N_1G}(a,b) \leqslant M_{N_1H}(a,b)$,

In view of (1.7), (2.12) and (3.5), we can easily check that

$$A(a,b) + H(a,b) \leqslant N_1(a,b) + N_3(a,b) \leqslant N_1(a,b) + N_2(a,b). \tag{3.6}$$

In this section we shall improve the inequalities (1.7) and then compare with the inequalities (3.1)-(3.5). This refinement is based on the following lemma.

Lemma 3.1. *Let $f_1, f_2 : I \subset \mathbb{R}_+ \to \mathbb{R}$ be two convex functions satisfying the assumptions:*
(i) $f_1(1) = f'_1(1) = 0$, $f_2(1) = f'_2(1) = 0$;
(ii) f_1 and f_2 are twice differentiable in $\mathbb{R}_+$;
(iii) there exists the real constants α, β such that $0 \leqslant \alpha < \beta$ and

$$\alpha \leqslant \frac{f''_1(x)}{f''_2(x)} \leqslant \beta, \;\; f''_2(x) > 0, \tag{3.7}$$

for all $x > 0$ then we have the inequalities:

$$\alpha \; \phi_{f_2}(a,b) \leqslant \phi_{f_1}(a,b) \leqslant \beta \; \phi_{f_2}(a,b), \tag{3.8}$$

for all $a, b \in (0,\infty)$.

Proof. Let us consider the functions

$$k(x) = f_1(x) - \alpha\ f_2(x)$$

and

$$h(x) = \beta\ f_2(x) - f_1(x),$$

where α and $\beta$are as given by (3.7).

In view of item (i), we have $k(1) = h(1) = 0$ and $k'(1) = h'(1) = 0$. Since the functions $f_1(x)$ and $f_2(x)$ are twice differentiable, then in view of (3.7), we have

$$k''(x) = f_1''(x) - \alpha\ f_2''(x) = f_2''(x)\left(\frac{f_1''(x)}{f_2''(x)} - \alpha\right) \geqslant 0\ , \tag{3.9}$$

and

$$h''(x) = \beta\ f_2''(x) - f_1''(x) = f_2''(x)\left(M - \frac{f_1''(x)}{f_2''(x)}\right) \geqslant 0, \tag{3.10}$$

for all $x \in (0,\infty)$.

In view of (3.9) and (3.10), we can say that the functions $k(\cdot)$ and $h(\cdot)$, are convex on $I \subset \mathbb{R}_+$.

According to (2.14), we have

$$a\ k\left(\frac{b}{a}\right) = a\left[f_1\left(\frac{b}{a}\right) - \alpha\ f_2\left(\frac{b}{a}\right)\right] = a\ f_1\left(\frac{b}{a}\right) - \alpha a\ f_2\left(\frac{b}{a}\right) \geqslant 0, \tag{3.11}$$

and

$$a\ h\left(\frac{b}{a}\right) = a\left[\beta f_2\left(\frac{b}{a}\right) - f_1\left(\frac{b}{a}\right)\right] = \beta a\ f_2\left(\frac{b}{a}\right) - f_1\left(\frac{b}{a}\right) \geqslant 0, \tag{3.12}$$

Combining (3.11) and (3.12) we have the proof of (3.8). □

Theorem 3.1. *The following inequalities among the mean differences hold:*

$$M_{SA}(a,b) \leqslant \frac{1}{3}M_{SH}(a,b) \leqslant \frac{1}{2}M_{AH}(a,b) \leqslant \frac{1}{2}M_{SG}(a,b) \leqslant M_{AG}(a,b). \tag{3.13}$$

Proof. In order to prove the above theorem, we shall prove each part separately.

Let us consider

$$g_{SA\_SH}(x) = \frac{f''_{SA}(x)}{f''_{SH}(x)} = \frac{(x+1)^3}{(x+1)^3 + 4\sqrt{2}(x^2+1)^{3/2}},\ x \in (0,\infty),$$

This gives

$$g'_{SA\_SH}(x) = -\frac{24(x-1)(x^2+1)(x+1)^2}{\sqrt{2(x^2+1)}\left[(x+1)^3 + 4\sqrt{2}(x^2+1)^{3/2}\right]^2}\begin{cases} \geqslant 0, & x \leqslant 1 \\ \leqslant 0, & x \geqslant 1 \end{cases}. \tag{3.14}$$

In view of (3.14) we conclude that the function $g_{SA\_SH}(x)$ increasing in $x \in (0,1)$ and decreasing in $x \in (1,\infty)$, and hence

$$\beta = \sup_{x\in(0,\infty)} g_{SA\_SH}(x) = g_{SA\_SH}(1) = \frac{1}{3}. \tag{3.15}$$

Applying (3.8) for the *difference of means* $M_{SA}(a,b)$ and $M_{SH}(a,b)$, and using (3.15), we get

$$M_{SA}(a,b) \leqslant \frac{1}{3}M_{SH}(a,b). \tag{3.16}$$

Let us consider

$$g_{SH\_AH}(x)=\frac{f''_{SH}(x)}{f''_{AH}(x)}=\frac{(x+1)^3+4\sqrt{2}(x^2+1)^{3/2}}{4\sqrt{2}(x^2+1)^{3/2}},\ x\in(0,\infty),$$

This gives

$$g'_{SH\_AH}(x)=-\frac{3(x-1)(x+1)^2}{4\sqrt{2}(x^2+1)^{5/2}}\begin{cases}\geqslant 0, & x\leqslant 1\\ \leqslant 0, & x\geqslant 1\end{cases}. \tag{3.17}$$

In view of (3.17), we conclude that the function $g_{SH\_AH}(x)$ is increasing in $x\in(0,1)$ and decreasing in $x\in(1,\infty)$, and hence

$$\beta=\sup_{x\in(0,\infty)} g_{SH\_AH}(x)=g_{SG\_AH}(1)=\frac{3}{2}. \tag{3.18}$$

Applying (3.8) for the *difference of means* $M_{SH}(a,b)$ and $M_{AH}(a,b)$, and using (3.18), we get

$$M_{SH}(a,b)\leqslant\frac{3}{2}M_{AH}(a,b). \tag{3.19}$$

Let us consider

$$g_{SG\_AH}(x)=\frac{f''_{SG}(x)}{f''_{AH}(x)}=\frac{(x+1)^3\left[4x^{3/2}+\sqrt{2}\ (x^2+1)^{3/2}\right]}{16\sqrt{2}\ (x^2+1)^{3/2}x^{3/2}},\ x\in(0,\infty),$$

This gives

$$g'_{SG\_AH}(x)=\frac{3(x+1)^4(x-1)\left[\sqrt{2}\,(x^2+1)^{5/2}-8x^{5/2}\right]}{32\sqrt{2}\,[x(x+1)]^{5/2}}\begin{cases}\geqslant 0, & x\geqslant 1,\\ \leqslant 0, & x\leqslant 1,\end{cases} \tag{3.20}$$

where we have used the fact that $x^2+1\geqslant 2x$, $\forall x\in(0,\infty)$.

In view of (3.20), we conclude that the function $g_{SG\_AH}(x)$ is decreasing in $x\in(0,1)$ and increasing in $x\in(1,\infty)$, and hence

$$\alpha=\inf_{x\in(0,\infty)} g_{SG\_AH}(x)=g_{SG\_AH}(1)=1. \tag{3.21}$$

Applying (3.8) for the *difference of means* $M_{AH}(a,b)$ and $M_{SG}(a,b)$, and using (3.21), we get

$$M_{AH}(a,b)\leqslant M_{SG}(a,b). \tag{3.22}$$

Let us consider

$$g_{SG\_AG}(x)=\frac{f''_{SG}(x)}{f''_{AG}(x)}=\frac{4x^{3/2}+\sqrt{2}\ (x^2+1)^{3/2}}{\sqrt{2}\ (x^2+1)^{3/2}},\ x\in(0,\infty),$$

This gives

$$g'_{SG\_AG}(x)=-\frac{6(x-1)(x+1)\sqrt{x}}{\sqrt{2}(x^2+1)^{5/2}}\begin{cases}\geqslant 0, & x\leqslant 1\\ \leqslant 0, & x\geqslant 1\end{cases}. \tag{3.23}$$

In view of (3.23), we conclude that the function $g_{SG\_AG}(x)$ is increasing in $x\in(0,1)$ and decreasing in $x\in(1,\infty)$, and hence

$$M=\sup_{x\in(0,\infty)} g_{SG\_AG}(x)=g_{SG\_AG}(1)=2. \tag{3.24}$$

Applying (3.8) for the *difference of means* $M_{SG}(a,b)$ and $M_{AG}(a,b)$, and using (3.21), we get

$$\frac{1}{2}M_{SG}(a,b) \leqslant M_{AG}(a,b) \tag{3.25}$$

Combining the results (3.16), (3.19), (3.22) and (3.25) we get the proof of the inequality (3.13). □

Corollary 3.1. *The following inequalities hold:*

$$\begin{aligned} H(a,b) &\leqslant G(a,b) \leqslant \frac{2H(a,b)+S(a,b)}{3} \leqslant \frac{A(a,b)+H(a,b)}{2} \\ &\leqslant \frac{S(a,b)+G(a,b)}{2} \leqslant \frac{H(a,b)+2S(a,b)}{3} \\ &\leqslant A(a,b) \leqslant S(a,b)+H(a,b)-G(a,b) \\ &\leqslant S(a,b) \leqslant 3\left[A(a,b)-G(a,b)\right]+H(a,b). \end{aligned} \tag{3.26}$$

Proof. Simplifying the results given in (3.16), (3.19), (3.22) and (3.25) we get the required result. □

Remark 3.1. *The inequalities (3.26) are the improvement over the following well known result*

$$\min\{a,b\} \leqslant H(a,b) \leqslant G(a,b) \leqslant A(a,b) \leqslant S(a,b) \leqslant \max\{a,b\}. \tag{3.27}$$

In the following corollary, we shall give a further improvement over the inequalities (3.26).

Corollary 3.2. *The following inequalities hold:*

$$\begin{aligned} H(a,b) &\leqslant \frac{2A(a,b)H(a,b)}{A(a,b)+H(a,b)} \leqslant G(a,b) \leqslant \frac{2H(a,b)+S(a,b)}{3} \\ &\leqslant \frac{A(a,b)+H(a,b)}{2} \leqslant \sqrt{\frac{(A(a,b))^2+(H(a,b))^2}{2}} \leqslant \frac{S(a,b)+G(a,b)}{2} \\ &\leqslant \frac{H(a,b)+2S(a,b)}{3} \leqslant A(a,b) \leqslant S(a,b)+H(a,b)-G(a,b) \\ &\leqslant S(a,b) \leqslant 3\left[A(a,b)-G(a,b)\right]+H(a,b). \end{aligned} \tag{3.28}$$

Proof. Replace a by $A(a,b)$ and b by $H(a,b)$ in (3.27) we get

$$\begin{aligned} \min\{A(a,b),H(a,b)\} &\leqslant H\left(A(a,b),H(a,b)\right) \leqslant G\left(A(a,b),H(a,b)\right) \\ &\leqslant A\left(A(a,b),H(a,b)\right) \leqslant S\left(A(a,b),H(a,b)\right) \leqslant \max\{A(a,b),H(a,b)\}. \end{aligned}$$

This gives

$$\begin{aligned} H(a,b) &\leqslant \frac{2A(a,b)H(a,b)}{A(a,b)+H(a,b)} \leqslant G(a,b) \leqslant \frac{A(a,b)+H(a,b)}{2} \\ &\leqslant \sqrt{\frac{(A(a,b))^2+(H(a,b))^2}{2}} \leqslant A(a,b) \leqslant S(a,b) \end{aligned} \tag{3.29}$$

The inequality (3.29) gives a different kind of improvement over the inequality (3.27).

Let us consider

$$K(a,b) = \frac{S(a,b)+G(a,b)}{2} - \sqrt{\frac{A(a,b)^2+H(a,b)^2}{2}} \tag{3.30}$$
$$= \frac{\left(\frac{S(a,b)+G(a,b)}{2}\right)^2 - \frac{A(a,b)^2+H(a,b)^2}{2}}{\frac{S(a,b)+G(a,b)}{2} + \sqrt{\frac{A(a,b)^2+H(a,b)^2}{2}}}.$$

Now we shall show that

$$\left(\frac{S(a,b)+G(a,b)}{2}\right)^2 - \frac{A(a,b)^2+H(a,b)^2}{2} \geqslant 0. \tag{3.31}$$

For it, let us consider

$$k(x) = \left[\frac{\sqrt{2(x^2+1)}}{4} + \frac{\sqrt{x}}{2}\right]^2 - \frac{1}{2}\left[\left(\frac{x+1}{2}\right)^2 + \left(\frac{2x}{x+1}\right)^2\right] \tag{3.32}$$
$$= \frac{8x^2 + (x+1)^2\sqrt{2x(x^2+1)}}{4(x+1)^2} > 0, \ \forall x \in (0,\infty).$$

Now the expression (3.32) together with (2.13) give us (3.31), or equivalently, we can say that

$$\sqrt{\frac{A(a,b)^2+H(a,b)^2}{2}} \leqslant \frac{S(a,b)+G(a,b)}{2}. \tag{3.33}$$

Finally, the inequalities (3.26), (3.29) and (3.33) give us the proof of the inequalities (3.29). This completes the proof of the corollary. □

Theorem 3.2. *The following inequalities hold:*

$$\frac{1}{8}M_{AH}(a,b) \leqslant M_{N_2N_1}(a,b) \leqslant \frac{1}{3}M_{N_2G}(a,b) \leqslant \frac{1}{4}M_{AG}(a,b) \leqslant M_{AN_2}(a,b). \tag{3.34}$$

Proof. In order to prove the above theorem, we shall prove each part separately.

Let us consider

$$g_{AH\_N_2N_1}(x) = \frac{f''_{AH}(x)}{f''_{N_2N_1}(x)} = \frac{32x^{5/2}(2x+2)^{3/2}}{(x+1)^3\left[-2x-2x^{5/2}+x(2x+2)^{3/2}\right]}, \ x \in (0,\infty).$$

This gives

$$g'_{AH\_N_2N_1}(x) = -\frac{48\sqrt{2x(x+1)}}{(x+1)^4\left[-2x-2x^{5/2}+x(2x+2)^{3/2}\right]^2}\times$$
$$\times\left[4x^2(1-x^{5/2}) + x^2(x-1)(2x+2)^{5/2}\right]$$
$$= \frac{48x^2(x+1)\left(1-\sqrt{x}\right)\sqrt{2x(x+1)}}{(x+1)^4\left[-2x-2x^{5/2}+x(2x+2)^{3/2}\right]^2}\times$$
$$\times\left[\sqrt{2}\left(\sqrt{x}+1\right)(x+1)^{3/2} - \left(x^2+x^{3/2}+x+\sqrt{x}+1\right)\right].$$

Since $\sqrt{2(x+1)} \geqslant \sqrt{x}+1$, $\forall x \in (0,\infty)$, then this implies that

$$\sqrt{2}(x+1)^{3/2}\left(\sqrt{x}+1\right) \geqslant \left(\sqrt{x}+1\right)^2(x+1)$$
$$\geqslant x^2+x^{3/2}+x+\sqrt{x}+1$$

Thus we conclude that

$$g'_{AH\_N_2N_1}(x) \begin{cases} < 0, & x > 1 \\ > 0, & x < 1 \end{cases}. \tag{3.35}$$

In view of (3.35), we conclude that the function $g_{AH\_N_2N_1}(x)$ is increasing in $x \in (0,1)$ and decreasing in $x \in (1,\infty)$, and hence

$$\beta = \sup_{x\in(0,\infty)} g_{AH\_N_2N_1}(x) = g_{AH\_N_2N_1}(1) = 8. \tag{3.36}$$

Applying (3.8) for the *difference of means* $M_{AH}(a,b)$ and $M_{N_2N_1}(a,b)$ along with (3.36), we get

$$\frac{1}{8}M_{AH}(a,b) \leqslant M_{N_2N_1}(a,b) \tag{3.37}$$

Let us consider

$$g_{N_2N_1\_N_2G}(x) = \frac{f''_{N_2N_1}(x)}{f''_{N_2G}(x)} = \frac{-2x - 2x^{5/2} + x(2x+2)^{3/2}}{2x\left[1 + x^{3/2} - (2x+2)^{3/2}\right]},\ x \in (0,\infty).$$

This gives

$$g'_{N_2N_1\_N_2G_1}(x) = \frac{3x^2\sqrt{2x+2}\,(1-\sqrt{x})}{2x^2\left[-1 - x^{3/2} + (2x+2)^{3/2}\right]^2} \begin{cases} < 0, & x > 1, \\ > 0, & x < 1. \end{cases} \tag{3.38}$$

In view of (3.38), we conclude that the function $g_{N_2N_1\_N_2G}(x)$ is increasing in $x \in (0,1)$ and decreasing in $x \in (1,\infty)$, and hence

$$\beta = \sup_{x\in(0,\infty)} g_{N_2N_1\_N_2G}(x) = g_{N_2N_1\_N_2G}(1) = \frac{1}{3}. \tag{3.39}$$

Applying (3.8) for the *difference of means* $M_{N_2N_1}(a,b)$ and $M_{N_2G}(a,b)$ along with (3.39), we get

$$M_{N_2N_1}(a,b) \leqslant \frac{1}{3}M_{N_2G}(a,b). \tag{3.40}$$

Let us consider

$$g_{N_2G\_AG}(x) = \frac{f''_{N_2G}(x)}{f''_{AG}(x)} = -\frac{1 + x^{3/2} - (2x+2)^{3/2}}{(2x+2)^{3/2}},\ x \in (0,\infty).$$

This gives

$$g'_{N_2G\_AG}(x) = \frac{3\,(1-\sqrt{x})}{(2x+2)^{5/2}} \begin{cases} \leqslant 0, & x \geqslant 1 \\ \geqslant 0, & x \leqslant 1 \end{cases}. \tag{3.41}$$

In view of (3.41), we conclude that the function $g_{AH\_N_2N_1}(x)$ is increasing in $x \in (0,1)$ and decreasing in $x \in (1,\infty)$, and hence

$$\beta = \sup_{x\in(0,\infty)} g_{N_2G\_AG}(x) = g_{N_2G\_AG}(1) = \frac{3}{4}. \tag{3.42}$$

Applying (3.8) for the *difference of means* $M_{N_2G}(a,b)$ and $M_{AG}(a,b)$ along with (3.42) we get

$$M_{N_2G}(a,b) \leqslant \frac{3}{4}M_{AG}(a,b). \tag{3.43}$$

Let us consider

$$g_{AG\_AN_2}(x) = \frac{f''_{AG}(x)}{f''_{AN_2}(x)} = \frac{(2x+2)^{3/2}}{(\sqrt{x}+1)(x-\sqrt{x}+1)},\ x \in (0,\infty).$$

This gives

$$g'_{AG\_AN_2}(x) = \frac{3(1-\sqrt{x})\sqrt{2x+2}}{(\sqrt{x}+1)^2(x-\sqrt{x}+1)^2} \begin{cases} \leqslant 0, & x \geqslant 1 \\ \geqslant 0, & x \leqslant 1 \end{cases}. \tag{3.44}$$

In view of (3.44), we conclude that the function $g_{AG\_AN_2}(x)$ is increasing in $x \in (0,1)$ and decreasing in $x \in (1,\infty)$, and hence

$$\beta = \sup_{x\in(0,\infty)} g_{AG\_AN_2}(x) = g_{AG\_AN_2}(1) = 4. \tag{3.45}$$

Applying (3.8) for the *difference of means* $M_{AG}(a,b)$ and $M_{AN_2}(a,b)$ along with (3.45) we get the required result.

$$\frac{1}{4}M_{AG}(a,b) \leqslant M_{AN_2}(a,b). \tag{3.46}$$

Combining the results (3.37), (3.40), (3.43) and (3.46) we get the proof of the inequalities (3.34). □

Corollary 3.3. *The inequalities hold:*

$$\begin{aligned} H(a,b) &\leqslant G(a,b) \leqslant \frac{G(a,b)+H(a,b)+3N_2(a,b)}{5} \\ &\leqslant \frac{G(a,b)+2N_2(a,b)}{3} \leqslant N_1(a,b) \leqslant \frac{2A(a,b)+7N_1(a,b)}{9} \leqslant N_2(a,b) \\ &\leqslant \frac{A(a,b)+N_1(a,b)}{2} \leqslant \frac{7A(a,b)+H(a,b)}{8} \leqslant A(a,b). \end{aligned} \tag{3.47}$$

Proof. Follows in view of (3.32), (3.35), (3.38), (3.41) and (3.6). □

Remark 3.2. *The inequalities (3.47) can be considered as an improvement over the following inequalities:*

$$H(a,b) \leqslant G(a,b) \leqslant N_1(a,b) \leqslant N_2(a,b) \leqslant A(a,b). \tag{3.48}$$

Theorem 3.3. *The following inequalities hold:*

$$M_{SA}(a,b) \leqslant \frac{4}{5}M_{SN_2}(a,b) \leqslant 4M_{AN_2}(a,b), \tag{3.49}$$

$$M_{SH}(a,b) \leqslant 2M_{SN_1}(a,b) \leqslant \frac{3}{2}M_{SG}(a,b), \tag{3.50}$$

and

$$M_{SA}(a,b) \leqslant \frac{3}{4}M_{SN_3}(a,b) \leqslant \frac{2}{3}M_{SN_1}(a,b). \tag{3.51}$$

Proof. In order to prove the above theorem, we shall prove each part separately.
Let us consider

$$g_{SA\_SN_2}(x) = \frac{f''_{SA}(x)}{f''_{SN_2}(x)} = \frac{8x^{3/2}(2x+2)^{3/2}}{8x^{3/2}(2x+2)^{3/2}+(1+x^{3/2})(2x^{3/2}+2)^{3/2}},\ x \in (0,\infty).$$

This gives

$$g'_{SA\_SN_2}(x) = -\frac{96\sqrt{x(x^2+1)(x+1)}}{\left[8x^{3/2}(2x+2)^{3/2}+(1+x^{3/2})(2x^{3/2}+2)^{3/2}\right]^2}\times$$
$$\times\left[(x^2-1)(x^{5/2}+1)+2x(x^{5/2}-1)\right]$$
$$= -\frac{96\left(\sqrt{x}-1\right)\sqrt{x(x^2+1)(x+1)}}{\left[8x^{3/2}(2x+2)^{3/2}+(1+x^{3/2})(2x^{3/2}+2)^{3/2}\right]^2}\times$$
$$\times\left[\left(\sqrt{x}+1\right)(x+1)(x^{5/2}+1)+2x\left(x^2+x^{3/2}+x+\sqrt{x}+1\right)\right].$$

Thus, we have

$$g'_{SA\_SN_2}(x)\begin{cases}\geqslant 0, & x\geqslant 1,\\ \leqslant 0, & x\leqslant 1.\end{cases} \tag{3.52}$$

In view of (3.52), we conclude that the function $g_{SA\_SN_2}(x)$ is increasing in $x\in(0,1)$ and decreasing in $x\in(1,\infty)$, and hence

$$\beta=\sup_{x\in(0,\infty)} g_{SA\_SN_2}(x)=g_{SA\_SN_2}(1)=\frac{4}{5}. \tag{3.53}$$

Applying (3.8) for the *difference of means* $M_{SA}(a,b)$ and $M_{SN_2}(a,b)$ along with (3.53) we get

$$M_{SA}(a,b)\leqslant\frac{4}{5}M_{SN_2}(a,b). \tag{3.54}$$

Let us consider

$$g_{SN_2\_AN_2}(x)=\frac{f''_{SN_2}(x)}{f''_{AN_2}(x)}=\frac{8x^{3/2}(2x+2)^{3/2}+(1+x^{3/2})(2x^2+2)^{3/2}}{(2x^2+2)(x^{3/2}+1)},\ x\in(0,\infty),$$

This gives

$$g'_{SN_2\_AN_2}(x) = -\frac{12\left[x(x+1)\right]^{9/2}\left[(x^2-1)(1+x^{5/2})+2x(x^{5/2}-1)\right]}{(x^2+1)^{5/2}x^4(x+1)^4(x^{3/2}+1)^2}$$
$$= -\frac{12\left(x(x+1)\right)^{9/2}\left(\sqrt{x}-1\right)}{(x^2+1)^{5/2}x^4(x+1)^4(x^{3/2}+1)^2}\times$$
$$\times\left[\left(\sqrt{x}+1\right)(x+1)(x^{5/2}+1)+2x\left(x^2+x^{3/2}+x+\sqrt{x}+1\right)\right].$$

Thus we have

$$g'_{SN_2\_AN_2}(x)\begin{cases}\geqslant 0, & x\leqslant 1,\\ \leqslant 0, & x\geqslant 1.\end{cases} \tag{3.55}$$

In view of (3.55), we conclude that the function $g_{SN_2\_AN_2}(x)$ is increasing in $x\in(0,1)$ and decreasing in $x\in(1,\infty)$, and hence

$$\beta=\sup_{x\in(0,\infty)} g_{SN_2\_AN_2}(x)=g_{SN_2\_AN_2}(1)=\frac{4}{5}. \tag{3.56}$$

Applying (3.8) for the *difference of means* $M_{SN_2}(a,b)$ and $M_{AN_2}(a,b)$ along with (3.56) we get

$$\frac{1}{5}M_{SN_2}(a,b)\leqslant M_{AN_2}(a,b). \tag{3.57}$$

Combining the results (3.54) and (3.57) we get the proof of the inequalities (3.49). Now we shall give the proof of (3.50).

Let us consider

$$g_{SH\_SN_1}(x) = \frac{f''_{SH}(x)}{f''_{SN_1}(x)} = \frac{16x^{3/2}\left[(x+1)^3 + 2(2x^2+2)^{3/2}\right]}{(x+1)^3\left[16x^{3/2} + (2x^2+2)^{3/2}\right]},\ x \in (0,\infty),$$

This gives

$$\begin{aligned} g'_{SH\_SN_1}(x) &= -\frac{48\sqrt{2x^2+2}}{x^2(x+1)^4\left[16x^{3/2} + (2x^2+2)^{3/2}\right]^2}\times \\ &\quad \times \left[64x^{9/2}(1-x) + 5x^4(x^2-1) + 4x^3(x^4-1)\right. \\ &\qquad \left. + x^2(x^2-1) + x^2(x-1)(2x^2+2)^{5/2}\right] \\ &= -\frac{1536x^2(x-1)\sqrt{2x^2+2}}{x^2(x+1)^4\left[16x^{3/2} + (2x^2+2)^{3/2}\right]^2}\times \\ &\quad \times \left\{\left[\left(\frac{x+1}{2}\right)^5 - \left(\sqrt{x}\right)^5\right] + \left[\left(\sqrt{\frac{x^2+1}{2}}\right)^5 - \left(\sqrt{x}\right)^5\right]\right\}. \end{aligned}$$

Since $S(a,b) \geqslant A(a,b) \geqslant G(a,b)$, one gets

$$g'_{SH\_SN_1}(x) \begin{cases} \geqslant 0, & x \leqslant 1, \\ \leqslant 0, & x \geqslant 1. \end{cases} \tag{3.58}$$

In view of (3.58) we conclude that the function $g_{SH\_SN_1}(x)$ increasing in $x \in (0,1)$ and decreasing in $x \in (1,\infty)$, and hence

$$\beta = \sup_{x\in(0,\infty)} g_{SH\_SN_1}(x) = g_{SH\_SN_1}(1) = 2. \tag{3.59}$$

Applying (3.8) for the *difference of means* $M_{SH}(a,b)$ and $M_{SN_1}(a,b)$ along with (3.59) we get

$$M_{SH}(a,b) \leqslant 2M_{SN_1}(a,b). \tag{3.60}$$

Let us consider

$$g_{SN_1\_SG}(x) = \frac{f''_{SN_1}(x)}{f''_{SG}(x)} = \frac{8x^{3/2} + (x^2+1)\sqrt{2x^2+2}}{2\left[4x^{3/2} + (x^2+1)\sqrt{2x^2+2}\right]},\ x \in (0,\infty),$$

This gives

$$g'_{SN_1\_SG}(x) = -\frac{3(x^2-1)\sqrt{2x^3+2x}}{\left[4x^{3/2} + (x^2+1)\sqrt{2x^2+2}\right]^2} \begin{cases} \geqslant 0, & x \leqslant 1, \\ \leqslant 0, & x \geqslant 1. \end{cases} \tag{3.61}$$

In view of (3.61), we conclude that the function $g_{SN_1\_SG}(x)$ is increasing in $x \in (0,1)$ and decreasing in $x \in (1,\infty)$, and hence

$$\beta = \sup_{x\in(0,\infty)} g_{SN_1\_SG}(x) = g_{SN_1\_SG}(1) = \frac{3}{4}. \tag{3.62}$$

Applying (3.8) for the *difference of means* $M_{SN_1}(a,b)$ and $M_{SG}(a,b)$ along with (3.62) we get

$$M_{SN_1}(a,b) \leqslant \frac{3}{4} M_{SG}(a,b). \tag{3.63}$$

Combining the results given in (3.60) and (3.63) we get the proof of the inequalities (3.50). Let us prove now the inequalities (3.51).

Let us consider

$$g_{SA\_SN_3}(x) = \frac{f''_{SA}(x)}{f''_{SN_3}(x)} = \frac{24x^{3/2}}{24x^{3/2} + (2x^2+2)^{3/2}},\ x \in (0,\infty),$$

This gives

$$g'_{SA\_SN_3}(x) = -\frac{72(x-1)(x+1)\sqrt{2x(x^2+1)}}{\left[24x^{3/2} + (2x^2+2)^{3/2}\right]^2} \begin{cases} \geqslant 0, & x \leqslant 1, \\ \leqslant 0, & x \geqslant 1. \end{cases} \tag{3.64}$$

In view of (3.64), we conclude that the function $g_{SA\_SN_3}(x)$ is increasing in $x \in (0,1)$ and decreasing in $x \in (1,\infty)$, and hence

$$\beta = \sup_{x\in(0,\infty)} g_{SA\_SN_3}(x) = g_{SA\_SN_3}(1) = \frac{3}{4}. \tag{3.65}$$

Applying (3.8) for the *difference of means* $M_{SA}(a,b)$ and $M_{SN_3}(a,b)$ along with (3.65) we get

$$M_{SA}(a,b) \leqslant \frac{3}{4} M_{SN_3}(a,b). \tag{3.66}$$

Let us consider

$$g_{SN_3\_SN_1}(x) = \frac{f''_{SN_3}(x)}{f''_{SN_1}(x)} = \frac{2\left[24x^{3/2} + (2x^2+2)^{3/2}\right]}{3\left[16x^{3/2} + (2x^2+2)^{3/2}\right]},\ x \in (0,\infty),$$

This gives

$$g'_{SN_3\_SN_1}(x) = -\frac{(x-1)(x+1)\sqrt{2x(x^2+1)}}{\left[16x^{3/2} + (2x^2+2)^{3/2}\right]^2} \begin{cases} \geqslant 0, & x \leqslant 1, \\ \leqslant 0, & x \geqslant 1. \end{cases} \tag{3.67}$$

In view of (3.67), we conclude that the function $g_{SN_3\_SN_1}(x)$ is increasing in $x \in (0,1)$ and decreasing in $x \in (1,\infty)$, and hence

$$\beta = \sup_{x\in(0,\infty)} g_{SN_3\_SN_1}(x) = g_{SN_3\_SN_1}(1) = \frac{3}{4}. \tag{3.68}$$

Applying (3.8) for the *difference of means* $M_{SN_3}(a,b)$ and $M_{SN_1}(a,b)$ along with (3.68) we get

$$M_{SN_3}(a,b) \leqslant \frac{8}{9} M_{SN_1}(a,b). \tag{3.69}$$

Combining the results given in (3.66) and (3.69) we get the proof of the inequalities (3.54). This completes the proof of the theorem. □

Corollary 3.4. *The following inequalities hold:*

$$\begin{aligned} G(a,b) &\leqslant \frac{S(a,b)+3G(a,b)}{4} \leqslant N_1(a,b) \leqslant \frac{S(a,b)+8N_1(a,b)}{9} \\ &\leqslant N_3(a,b) \leqslant N_2(a,b) \leqslant \frac{A(a,b)+N_1(a,b)}{2} \leqslant \frac{S(a,b)+2N_1(a,b)}{3} \\ &\leqslant \left(\frac{S(a,b)+4N_2(a,b)}{5}\right) \ or \ \left(\frac{S(a,b)+3N_3(a,b)}{4}\right) \leqslant A(a,b) \end{aligned} \tag{3.70}$$

and

$$G(a,b) \leqslant \frac{S(a,b)+2H(a,b)}{2} \leqslant N_1(a,b) \leqslant \frac{S(a,b)+H(a,b)}{2} \leqslant N_2(a,b). \tag{3.71}$$

Proof. The inequalities (3.49)-(3.51) lead us to (3.70) and (3.71). □

Remark 3.3. *The inequalities (3.70) can be considered as refinement over the inequality (1.7). Thus we have three different kind of refinements given by (3.28), (3.47) and (3.70) for the inequality (1.7). The inequalities (3.71) gives alternative improvement among the means $G(a,b)$, $N_1(a,b)$ and $N_2(a,b)$.*

References

[1] E.F. BECKENBACH and R. BELLMAN, Inequalities, Springer-Verlag, New York, 1971.

[2] J. BORWEIN and P. BORWEIN, π *and the AGM*, Wiley, 1987.

[3] S.S. DRAGOMIR and C.E.M. PEARCE, Selected Topics on Hermite-Hadamard Inequalities and Applications, Research Report Collection, Monograph, 2002, available on line: rgmia.vu.edu.au/monographs/index.html

[4] I.J. TANEJA, On a Difference of Jensen Inequality and its Applications to Mean Divergence Measures – *RGMIA Research Report Collection*, *http://rgmia.vu.edu.au*, **7**(4)(2004), Art. 16. Also in:arXiv:math.PR/0501302 v1 19 Jan 2005.

[5] I.J. TANEJA, On Symmetric and Non-Symmetric Divergence Measures and Their Generalizations, to appear as a chapter in: Advances in Imaging and Electron Physics, 2005.

[6] ZHI-HUA ZHANG and YU-DONG WU, The New Bounds of the Logarithmic Mean, *RGMIA Research Report Collection*, *http://rgmia.vu.edu.au*, **7**(2)(2004), Art. 7.

E-mail address: `taneja@mtm.ufsc.br`
URL: `http://www.mtm.ufsc.br/~taneja`

Inder Jeet Taneja, Departamento de Matemática, Universidade Federal de Santa Catarina, 88.040-900 Florianópolis, SC, Brazil