TWO LOGARITHMICALLY COMPLETELY MONOTONIC FUNCTIONS CONNECTED WITH GAMMA FUNCTION

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ABSTRACT. In this paper, the logarithmically complete monotonicity results of the functions $[\Gamma(1+x)]^y/\Gamma(1+xy)$ and $\Gamma(1+y)[\Gamma(1+x)]^y/\Gamma(1+xy)$ are established.

1. INTRODUCTION

In [3], the authors presented and proved, by using a geometrical method, the following double inequality

$$\frac{1}{n!} \leq \frac{[\Gamma(1+x)]^n}{\Gamma(1+nx)} \leq 1 \tag{1}$$

for $x \in [0,1]$ and $n \in \mathbb{N}$.

In [14], the author showed by analytical arguments that inequality (1) is an immediate consequence of the following monotonic property: For all $y \geq 1$, the function

$$f(x,y) = \frac{[\Gamma(1+x)]^y}{\Gamma(1+xy)} \tag{2}$$

is a decreasing function of $x \geq 0$. This monotonicity result leads to the following double inequality

$$\frac{1}{\Gamma(1+y)} \leq \frac{[\Gamma(1+x)]^y}{\Gamma(1+xy)} \leq 1 \tag{3}$$

for all $y \geq 1$ and $x \in [0,1]$, which is a generalization of inequality (1).

The purpose of this paper is to generalize the decreasingly monotonicity by J. Sándor in [14] to logarithmically complete monotonicity. Our main results are as follows.

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Theorem 1. *For given $y > 1$, the function $f(x, y)$ defined by (2) is decreasing and logarithmically concave with respect to $x \in (0, \infty)$, and the second order derivative of $-\ln f(x, y)$ with respect to x is completely monotonic in $x \in (0, \infty)$.*

For given $0 < y < 1$, the function $f(x, y)$ is increasing and logarithmically convex with respect to $x \in (0, \infty)$, and the second order derivative of $\ln f(x, y)$ with respect to x is completely monotonic in $x \in (0, \infty)$.

For given $x \in (0, \infty)$, the function $f(x, y)$ is logarithmically concave with respect to $y \in (0, \infty)$, and the first order derivative of $-\ln f(x, y)$ with respect to y is completely monotonic in $y \in (0, \infty)$.

Theorem 2. *For given $x \in (0, \infty)$, let*

$$F_x(y) = \frac{\Gamma(1 + y)[\Gamma(1 + x)]^y}{\Gamma(1 + xy)} \tag{4}$$

in $\in (0, \infty)$. If $0 < x < 1$ then the second order derivative of $\ln F_x(y)$ is completely monotonic in $(0, \infty)$, if $x > 1$ then the second order derivative of $-\ln F_x(y)$ is completely monotonic in $(0, \infty)$.

2. Definitions and Lemmas

Recall that the definition of completely monotonic functions is well-known.

Definition 1. A function f is called completely monotonic on an interval I if f has derivatives of all orders on I and

$$0 \le (-1)^k f^{(k)}(x) < \infty \tag{5}$$

for all $k \ge 0$ on I.

The class of completely monotonic functions on I is denoted by $\mathcal{C}[I]$.

In 2004, the paper [9] explicitly introduces the following notion or terminology.

Definition 2. A positive function f is called logarithmically completely monotonic on an interval I if f has derivatives of all orders on I and its logarithm $\ln f$ satisfies

$$0 \le (-1)^k [\ln f(x)]^{(k)} < \infty \tag{6}$$

for all $k \in \mathbb{N}$ on I.

The set of logarithmically completely monotonic functions on an interval I is denoted by $\mathcal{L}[I]$.

Among other things, it is proved in [8, 9, 15] that a logarithmically completely monotonic function is always completely monotonic, that is, $\mathcal{L}[I] \subset \mathcal{C}[I]$, but not conversely. Motivated by the papers [9, 13], among other things, it is further revealed in [4] that $\mathcal{S}\backslash\{0\} \subset \mathcal{L}[(0,\infty)] \subset \mathcal{C}[(0,\infty)]$, where $\mathcal{S}$ denotes the set of Stieltjes transforms. In [4, Theorem 1.1] and [5, 12] it is pointed out that the logarithmically completely monotonic functions on $(0,\infty)$ can be characterized as the infinitely divisible completely monotonic functions studied by Horn in [6, Theorem 4.4]. In [10], among other things, a basic property of the logarithmically completely monotonic functions is obtained: If $h'(x) \in \mathcal{C}[I]$ and $f(x) \in \mathcal{L}[h(I)]$, then $f\big(h(x)\big) \in \mathcal{L}[I]$. For more information on the logarithmically completely monotonic functions defined by Definition 2, please refer to [4, 5, 8, 11, 12, 13], especially [7, 10, 15], and the references therein.

The classical Euler gamma function $\Gamma(x)$ is defined for $x > 0$ by

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} \,\mathrm{d}\, t. \tag{7}$$

The logarithmic derivative of $\Gamma(x)$, denoted by $\psi(x) = \Gamma'(x)/\Gamma(x)$, is called psi or digamma function.

Lemma 1 ([2, 16, 17])**.** *For $x > 0$ and $r > 0$,*

$$\frac{1}{x^r} = \frac{1}{\Gamma(r)} \int_0^\infty t^{r-1} e^{-xt} \,\mathrm{d}\, t. \tag{8}$$

Lemma 2 ([2, 16, 17])**.** *The polygamma functions $\psi^{(k)}(x)$ can be expressed for $x > 0$ and $k \in \mathbb{N}$ as*

$$\psi^{(k)}(x) = (-1)^{k+1} \int_0^\infty \frac{t^k e^{-xt}}{1 - e^{-t}} \,\mathrm{d}\, t. \tag{9}$$

Formula (9) means that the psi function $\psi(x)$ is increasing, the polygamma functions $\psi^{(2k)}(x)$ are negative and increasing, and the polygamma functions $\psi^{(2k-1)}(x)$ are positive and decreasing in $(0,\infty)$ for $k \in \mathbb{N}$.

Lemma 3 ([1, p. 153])**.** *For $k \in \mathbb{N}$, as $x \to \infty$,*

$$|\psi^{(k)}(x)| \sim \frac{(k-1)!}{x^k}. \tag{10}$$

Lemma 4 ([18]). *Let $f_i(t)$ for $i=1,2$ be piecewise continuous in arbitrary finite intervals included in $(0,\infty)$, suppose there exist some constants $M_i>0$ and $c_i\ge 0$ such that $|f_i(t)|\le M_ie^{c_it}$ for $i=1,2$. Then*

$$\int_0^\infty\biggl[\int_0^t f_1(u)f_2(t-u)\,\mathrm{d}\,u\biggr]e^{-st}\,\mathrm{d}\,t=\int_0^\infty f_1(u)e^{-su}\,\mathrm{d}\,u\int_0^\infty f_2(v)e^{-sv}\,\mathrm{d}\,v. \tag{11}$$

Remark 1. Lemma 4 is the convolution theorem of Laplace transforms. It can be looked up in standard textbooks of integral transforms.

Lemma 5. *Let $i\in\mathbb{N}$ and $\alpha\in\mathbb{R}$. Then the functions $x^\alpha|\psi^{(i)}(1+x)|$ are strictly increasing in $(0,\infty)$ if and only if $\alpha\ge i$. In particular, the functions $x^{2i}\psi^{(2i)}(1+x)$ and $x^{2i+1}\psi^{(2i)}(1+x)$ are decreasing and the functions $x^{2i-1}\psi^{(2i-1)}(1+x)$ and $x^{2i}\psi^{(2i-1)}(1+x)$ are increasing in $[0,\infty)$.*

Proof. Let $g_\alpha(x)=x^\alpha|\psi^{(i)}(1+x)|$ for $i\in\mathbb{N}$. Differentiating $g_\alpha(x)$ and applying (8) and (9) yields

$$\begin{aligned}\frac{g_\alpha'(x)}{x^\alpha}&=\frac{\alpha}{x}|\psi^{(i)}(1+x)|-|\psi^{(i+1)}(1+x)|\\&=\alpha\int_0^\infty e^{-xt}\,\mathrm{d}\,t\int_0^\infty e^{-(x+1)t}\frac{t^i}{1-e^{-t}}\,\mathrm{d}\,t-\int_0^\infty e^{-(x+1)t}\frac{t^{i+1}}{1-e^{-t}}\,\mathrm{d}\,t.\end{aligned} \tag{12}$$

Using Lemma 4 leads to

$$\frac{g_\alpha'(x)}{x^\alpha}=\int_0^\infty e^{-xt}h_\alpha(t)\,\mathrm{d}\,t, \tag{13}$$

where

$$h_\alpha(t)=\alpha\int_0^t\frac{s^ie^{-s}}{1-e^{-s}}\,\mathrm{d}\,s-\frac{t^{i+1}e^{-t}}{1-e^{-t}}. \tag{14}$$

A simple calculation gives

$$p_\alpha(t)\triangleq e^{2t}\bigl(1-e^{-t}\bigr)^2t^{-i}h_\alpha'(t)=\bigl(e^t-1\bigr)(\alpha-i-1+t)+t. \tag{15}$$

It is clear that $p_\alpha(t)>0$ in $(0,\infty)$ is equivalent with

$$\alpha-i-1>\frac{te^t}{1-e^t}\triangleq q(t) \tag{16}$$

in $(0,\infty)$. It is easy to see that the function $q(t)$ is decreasing in $(0,\infty)$ and $\lim_{t\to0+}q(t)=-1$. Thus, if $\alpha\ge i$ then $p_\alpha(t)>0$ and $h_\alpha'(t)>0$ in $(0,\infty)$. From that $h_\alpha(t)$ is increasing and $\lim_{t\to0+}h_\alpha(t)=0$, it is obtained that $h_\alpha(t)>0$ in $(0,\infty)$, which implies that $g_\alpha'(x)>0$ and $g_\alpha(x)$ is strictly increasing for $x\in(0,\infty)$.

Assume the function $g_\alpha(x)$ is strictly increasing in $(0,\infty)$, then for $x\in(0,\infty)$

$$x^{i+1-\alpha}g_\alpha'(x)=\alpha x^i|\psi^{(i)}(1+x)|-x^{i+1}|\psi^{(i+1)}(1+x)|\ge 0. \tag{17}$$

Applying the asymptotic formula (10) we obtain

$$\lim_{x\to\infty}x^{i+1-\alpha}g_\alpha'(x)=(i-1)!(\alpha-i). \tag{18}$$

From (17) and (18) it follows that $\alpha\ge i$. □

3. Proofs of theorems

Proof of Theorem 1. Taking the logarithm of $f(x,y)$ and differentiating with respect to x for $k\in\mathbb{N}$ yields

$$\ln f(x,y)=y\ln\Gamma(1+x)-\ln\Gamma(1+xy), \tag{19}$$

$$\begin{aligned}\frac{\mathrm{d}^k[\ln f(x,y)]}{\mathrm{d}x^k}&=y\bigl[\psi^{(k-1)}(1+x)-y^{k-1}\psi^{(k-1)}(1+xy)\bigr]\\&=\frac{y}{x^{k-1}}\bigl[x^{k-1}\psi^{(k-1)}(1+x)-(xy)^{k-1}\psi^{(k-1)}(1+xy)\bigr],\end{aligned} \tag{20}$$

$$\frac{\mathrm{d}[\ln f(x,y)]}{\mathrm{d}y}=\ln\Gamma(1+x)-x\psi(1+xy), \tag{21}$$

$$\frac{\mathrm{d}^{k+1}[\ln f(x,y)]}{\mathrm{d}y^{k+1}}=-x^{k+1}\psi^{(k)}(1+xy). \tag{22}$$

By using Lemma 5, from (20) it is obtained for $i\in\mathbb{N}$ that

$$\frac{\mathrm{d}^{2i}[\ln f(x,y)]}{\mathrm{d}x^{2i}}\begin{cases}>0, & 0<y<1,\\<0, & y>1,\end{cases} \tag{23}$$

$$\frac{\mathrm{d}^{2i+1}[\ln f(x,y)]}{\mathrm{d}x^{2i+1}}\begin{cases}<0, & 0<y<1,\\>0, & y>1.\end{cases} \tag{24}$$

Since $\psi(x)$ is increasing in $(0,\infty)$, the first derivative

$$\frac{\mathrm{d}[\ln f(x,y)]}{\mathrm{d}x}\begin{cases}>0, & 0<y<1,\\<0, & y>1.\end{cases} \tag{25}$$

For $i\in\mathbb{N}$, from (9) it is deduced that

$$(-1)^i\frac{\mathrm{d}^{i+1}[\ln f(x,y)]}{\mathrm{d}y^{i+1}}>0 \tag{26}$$

in $(0,\infty)$. This implies $\mathrm{d}[\ln f(x,y)]/\mathrm{d}y$ is a decreasing function of $y\in(0,\infty)$. □

Proof of Theorem 2. Taking the logarithm of $F_x(y)$ and differentiating gives

$$\ln F_x(y) = \ln\Gamma(1+y) + y\ln\Gamma(1+x) - \ln\Gamma(1+xy), \tag{27}$$

$$[\ln F_x(y)]' = \psi(1+y) + \ln\Gamma(1+x) - x\psi(1+xy), \tag{28}$$

$$\begin{aligned}[\ln F_x(y)]^{(i+1)} &= \psi^{(i)}(1+y) - x^{i+1}\psi^{(i)}(1+xy)\\ &= \frac{1}{y^{i+1}}\bigl[y^{i+1}\psi^{(i)}(1+y) - (xy)^{i+1}\psi^{(i)}(1+xy)\bigr],\end{aligned} \tag{29}$$

where $i \in \mathbb{N}$.

For $i \in \mathbb{N}$, using Lemma 5 yields

$$[\ln F_x(y)]^{(2i+1)} \begin{cases} < 0, & 0 < x < 1, \\ > 0, & x > 1, \end{cases} \tag{30}$$

$$[\ln F_x(y)]^{(2i)} \begin{cases} > 0, & 0 < x < 1, \\ < 0, & x > 1. \end{cases} \tag{31}$$

This is equivalent to

$$(-1)^k[\ln F_x(y)]^{(k)} \begin{cases} > 0, & 0 < x < 1 \\ < 0, & x > 1 \end{cases} \tag{32}$$

for $k \geq 2$. The proof is complete. □

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