CERTAIN LOGARITHMICALLY N-ALTERNATING MONOTONIC FUNCTIONS INVOLVING GAMMA AND q-GAMMA FUNCTIONS

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ABSTRACT. In the paper, three basic properties of the logarithmically N-alternating monotonic functions are established and the monotonicity results of some functions involving the gamma and q-gamma functions, which are obtained in [W. E. Clark and M. E. H. Ismail, *Inequalities involving gamma and psi functions*, Anal. Appl. (Singap.) **1** (2003), no. 1, 129–140.], are generalized to the logarithmically N-alternating monotonicity.

1. INTRODUCTION

Recall that the definition of completely monotonic functions is well-known, and can be stated as follows.

Definition 1. A function f is called *completely monotonic* on an interval I if f has derivatives of all orders on I and

$$0 \le (-1)^k f^{(k)}(x) < \infty \tag{1}$$

for all $k \ge 0$ on I.

The class of completely monotonic functions on I is denoted by $\mathcal{C}[I]$.

In 2004, the paper [15] explicitly introduces the following notion or terminology.

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Definition 2. A positive function f is called *logarithmically completely monotonic* on an interval I if f has derivatives of all orders on I and its logarithm $\ln f$ satisfies

$$0 \le (-1)^k [\ln f(x)]^{(k)} < \infty \tag{2}$$

for all $k \in \mathbb{N}$ on I.

The set of logarithmically completely monotonic functions on an interval I is denoted by $\mathcal{L}[I]$.

Among other things, it is proved in [14, 15, 22] that a logarithmically completely monotonic function is always completely monotonic, that is, $\mathcal{L}[I] \subset \mathcal{C}[I]$, but not conversely. Motivated by the papers [15, 19], among other things, it is further revealed in [3] that $\mathcal{S}\backslash\{0\} \subset \mathcal{L}[(0,\infty)] \subset \mathcal{C}[(0,\infty)]$, where $\mathcal{S}$ denotes the set of Stieltjes transforms. In [3, Theorem 1.1] and [8, 18] it is pointed out that logarithmically completely monotonic functions on $(0,\infty)$ can be characterized as the infinitely divisible completely monotonic functions studied by Horn in [9, Theorem 4.4]. In [16], among other things, a basic property of the logarithmically completely monotonic functions is obtained: If $h'(x) \in \mathcal{C}[I]$ and $f(x) \in \mathcal{L}[h(I)]$, then $f\big(h(x)\big) \in \mathcal{L}[I]$. For more information on the logarithmically completely monotonic functions defined by Definition 2, please refer to [3, 8, 14, 17, 18, 19], especially [16, 22], and the references therein.

The following definition can be found in [6, 11, 12, 22].

Definition 3. A function f is called *N-alternating monotonic* on an interval I if there exists some nonnegative integer N such that inequality (1) holds for all $0 \le k \le N+1$ on I.

The class of N-alternating monotonic functions on an interval I will be denoted by $\mathcal{C}_{N+1}[I]$. Note that functions in $\mathcal{C}_N[I]$ are called "monotonic of order N" in [11, 12]. Here, we adopt the terminology "N-alternating monotonic" coined in [6]. It is obvious that $\mathcal{C}_\infty[I] \triangleq \lim_{N\to\infty} \mathcal{C}_N[I] = \mathcal{C}[I]$.

Further, by slightly modifying of corresponding classes of functions in [21, 22, 23] and formally assigning of names, we pose the following definitions.

Definition 4. A positive function f is said to be *logarithmically N-alternating monotonic* on an interval I if there exists some nonnegative integer N such that inequality (2) holds for all $1 \le k \le N+1$ on I.

Definition 5. For some nonnegative integer N, a function f is called *N-alternating monotonic to α-power* on an interval I if either $f \ge 0$ and $f^\alpha \in \mathcal{C}_{N+1}[I]$ for $\alpha > 0$ or $f > 0$ and $f^\alpha \in \mathcal{C}_{N+1}[I]$ for $\alpha < 0$. In particular, a positive function f is said to be *reciprocally N-alternating monotonic* on I if $1/f \in \mathcal{C}_{N+1}[I]$.

Definition 6. For some nonnegative integer N, a function f is said to be *completely monotonic to α-power* on an interval I if either $f \ge 0$ and $f^\alpha \in \mathcal{C}[I]$ for $\alpha > 0$ or $f > 0$ and $f^\alpha \in \mathcal{C}[I]$ for $\alpha < 0$. In particular, a positive function f is called *reciprocally completely monotonic* on I if $1/f \in \mathcal{C}[I]$.

The sets of logarithmically N-alternating monotonic functions, N-alternating monotonic functions to α-power and completely monotonic functions to α-power on an interval I are respectively denoted by $\mathcal{L}_{N+1}[I]$, $\mathcal{C}^\alpha_{N+1}[I]$ and $\mathcal{C}^\alpha[I]$. It is easy to see that $\mathcal{L}_\infty[I] \triangleq \lim_{N\to\infty} \mathcal{L}_N[I] = \mathcal{L}[I]$, $\mathcal{C}^\alpha_\infty[I] \triangleq \lim_{N\to\infty} \mathcal{C}^\alpha_{N+1}[I] = \mathcal{C}^\alpha[I]$.

In [20, 21, 22, 23] the following classes of functions are also defined:

$$\mathcal{D}^\alpha_N[I] = \{f(x) > 0 \mid [f^\alpha(x)]' \in \mathcal{C}_{N-1}[I], N \ge 1, \alpha < 0\}, \tag{3}$$

$$\mathcal{K}_N[I] = \{f(x) \mid f'(x) \in \mathcal{C}_{N-1}[I], N \ge 1\}, \tag{4}$$

$$\mathcal{D}^\alpha[I] = \mathcal{D}^\alpha_\infty[I] = \lim_{N\to\infty} \mathcal{D}^\alpha_N[I], \quad \alpha < 0, \tag{5}$$

$$\mathcal{K}[I] = \mathcal{K}_\infty[I] = \lim_{N\to\infty} \mathcal{K}_N[I], \tag{6}$$

$$\mathcal{T}\big[[0,\infty)\big] = \left\{ f(x) \,\middle|\, f(x) = \int_0^x \varphi(t)\,\mathrm{d}\,t < \infty, f(0) = 0, \varphi(t) \in \mathcal{C}[(0,\infty)] \right\}. \tag{7}$$

These classes of functions have the following inclusion relations for $N \in \mathbb{N} \cup \{\infty\}$:

$$\mathcal{D}^\alpha_1[I] = \mathcal{L}_1[I] \subset \mathcal{C}_1[I] = \mathcal{C}^1_1[I], \quad \alpha < 0, \tag{8}$$

$$\mathcal{C}^\alpha_N[I] \subset \mathcal{C}^{n\alpha}_N[I], \quad \alpha > 0, \quad n \in \mathbb{N}, \tag{9}$$

$$\mathcal{T}\big[[0,\infty)\big] \ne \mathcal{K}\big[[0,\infty)\big], \tag{10}$$

$$\mathcal{D}^\alpha_N[I] \subset \mathcal{D}^\beta_N[I], \quad \alpha < \beta < 0, \tag{11}$$

$$\mathcal{D}^\alpha_N[I] \subset \mathcal{C}^\beta_N[I], \quad \alpha < 0, \quad \beta > 0, \tag{12}$$

$$\mathcal{D}_N^{-\alpha}[I] \subset \mathcal{L}_N[I] \subset \mathcal{C}_N^{\alpha}[I], \quad \alpha > 0, \tag{13}$$

$$\mathcal{S} \subset \mathcal{D}^{-1}[(0,\infty)] \subset \mathcal{L}[(0,\infty)] \subset \mathcal{C}[(0,\infty)], \tag{14}$$

$$\mathcal{C}_{N+1}^{\alpha}[I] \subset \mathcal{C}_N^{\alpha}[I], \quad \alpha > 0, \tag{15}$$

$$\mathcal{D}_{N+1}^{\alpha}[I] \subset \mathcal{D}_N^{\alpha}[I], \quad \alpha < 0, \tag{16}$$

$$\mathcal{L}_{N+1}[I] \subset \mathcal{L}_N[I]. \tag{17}$$

Many basic properties of the classes of functions mentioned above were reproved, extended, collected, corrected and established in [22], among other things.

In Section 2 of this paper, we will prove the following results about the class $\mathcal{L}_N[I]$ of logarithmically N-alternating monotonic functions, analogies of them have recently been found for the class $\mathcal{L}[I]$ in [15, 16].

Theorem 1. *For $N \in \mathbb{N} \cup \{\infty\}$, if $h(x) \in \mathcal{K}_N[I]$ and $f \in \mathcal{L}_N[h(I)]$, then $f\big(h(x)\big) \in \mathcal{L}_N[I]$.*

Theorem 2. *Let $N \in \mathbb{N} \cup \{\infty\}$ and $f_i(x) \in \mathcal{L}_N[I]$ and $\alpha_i \ge 0$ for $1 \le i \le n$ with $n \in \mathbb{N}$. Then $\prod_{i=1}^n [f_i(x)]^{\alpha_i} \in \mathcal{L}_N[I]$.*

Theorem 3. *Let $N \in \mathbb{N}$ and $f(x) \in \mathcal{L}_N[I]$. Then $f(x)/f(x+\alpha) \in \mathcal{L}_{N-1}[J]$ if and only if $\alpha > 0$, where $J = I \cap \{x + \alpha \in I\}$.*

Let $r \ge 2$ be an integer. Canfield proved in [4] that the sequence $\binom{rm}{m}\sqrt{m}\,/c_1c_2^m$ is increasing with $m \ge 1$, where $c_1 = \sqrt{r/2\pi(r-1)}\,$, $c_2 = r^r/(r-1)^{r-1}$, and the quantity $c_1c_2^m/\sqrt{m}$ is the asymptotic value of $\binom{rm}{m}$. Motivated by Canfield's problem, Clark and Ismail obtained in [5] that the function

$$G(x) = \frac{\prod_{k=1}^n \Gamma(a_k x + 1)}{\Gamma(sx+1)(2\pi x)^{(n-1)/2}} \frac{s^{sx+1/2}}{\prod_{k=1}^n a_k^{a_k x + 1/2}} \tag{18}$$

is decreasing in $(0,\infty)$, where $a_i > 0$ for $1 \le i \le n$, $s = \sum_{k=1}^n a_k$, and $\Gamma(x)$ denotes the classical Euler gamma function defined by $\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}\,\mathrm{d}\,t$ for $\operatorname{Re} z > 0$. The gamma function $\Gamma(x)$, the psi or digamma function $\psi(x) = [\ln\Gamma(x)]' = \Gamma'(x)/\Gamma(x)$ and the polygamma functions $\psi^{(i)}(x)$ for $i \in \mathbb{N}$ are a class of the most important special functions [1, 24, 25] and have much extensive applications in many branches, for example, statistics, physics, engineering, and other mathematical sciences.

As a generalization of monotonicity result for the function $G(x)$, we shall show in Section 2 the following

Theorem 4. *Let $a_i > 0$ for $1 \le i \le n \in \mathbb{N}$ and $s = \sum_{k=1}^{n} a_k$. If $\sum_{i=1}^{n} a_i^k \ge s^k$ holds for some $k \in \mathbb{N}$, then $G(x) \in \mathcal{L}_k[(0,\infty)]$, that is, the function $G(x)$ defined by (18) is logarithmically k-alternating monotonic in the interval $(0,\infty)$.*

In [5] it is shown that the function

$$F_{a,b}(x) = \frac{[\Gamma(x+b)]^m}{x^{m/2}\Gamma(mx+a)} \tag{19}$$

is decreasing for $x \ge \max\{0, b-2, (b-2a)/(2m-3)\}$, where $b > a > 0$ and $m \ge 2$ is a positive integer.

As a generalization of the monotonicity result for the function $F_{a,b}(x)$, we shall show in Section 2 the following

Theorem 5. *Let a and b be positive numbers and*

$$\tau(a,b) = \inf_{u \in (0,1]} \{u^{-a} - u^{1-a} + 2u^{b-a}\}. \tag{20}$$

Further, let $m \ge 2$ and $k \in \mathbb{N}$ be positive integers and

$$\lambda(a,b,k,m) = \frac{(k-1)\ln m + \ln 2 - \ln \tau(a,b)}{m-1}. \tag{21}$$

Then $F_{a,b}(x) \in \mathcal{L}_k\big[\big(\lambda(a,b,k,m),\infty\big)\big] \cap \mathcal{L}_k[(0,\infty)]$. In particular,

$$F_{a,1}(x) \in \begin{cases} \mathcal{L}_k\left[\left[\dfrac{(k-1)\ln m}{m-1}, \infty\right)\right] & \text{for } a > \dfrac{1}{2}, \\ \mathcal{L}_k\left[\left[\dfrac{(k-1)\ln m + \ln 2 + \ln[a^a(1-a)^{1-a}]}{m-1}, \infty\right)\right] \\ \text{for } 0 < a \le \dfrac{1}{2}; \end{cases} \tag{22}$$

and

$$F_{a,b}(x) \in \begin{cases} \mathcal{L}_k\left[\left(\dfrac{(k-1)\ln m + \ln 2}{m-1}, \infty\right)\right] & \text{for } 0 < b < 1, \\ \mathcal{L}_k\left[\left[\dfrac{(k-1)\ln m + \ln 2 - \ln\left[1 + (1-b)\left(2b^b\right)^{1/(1-b)}\right]}{m-1}, \infty\right)\right] \\ \text{for } b > 1. \end{cases} \tag{23}$$

Recall the notation

$$(a;q)_m=\prod_{k=1}^{m}\left(1-aq^{k-1}\right) \tag{24}$$

for $m\in\mathbb{N}\cup\{\infty\}$ and that, when $0<q<1$, the q-gamma function is defined [2, 7] by

$$\Gamma_q(z)=(1-q)^{1-z}\prod_{i=0}^{\infty}\frac{1-q^{i+1}}{1-q^{z+i}}. \tag{25}$$

It is well known that q-gamma function is the q-analogue of the gamma function, that is, $\lim_{q\to1^-}\Gamma_q(z)=\Gamma(z)$.

Let $a_k>0$ for $1\le k\le n$ and $s=\sum_{i=1}^{n}a_k$. Define

$$H(x)=\frac{\prod_{k=1}^{n}\Gamma_q(a_kx+1)}{\Gamma_q(sx+1)[(q;q)_\infty]^{n-1}} \tag{26}$$

for $x\in(0,\infty)$. In [5] it was proved that the function $H(x)$ decreases to 1 on $(0,\infty)$.

As a generalization of this result, the following logarithmically N-alternating monotonic property for the function $H(x)$ defined by (26) is obtained.

Theorem 6. *Let $a_k>0$ for $1\le k\le n$ and $s=\sum_{i=1}^{n}a_k$. If $\sum_{i=1}^{n}a_i^k\ge s^k$ holds for some $k\in\mathbb{N}$, then $H(x)\in\mathcal{L}_k[(0,\infty)]$.*

2. Proofs of theorems

Proof of Theorem 1. Since $f\in\mathcal{L}_N[h(I)]$ is equivalent to $-f'/f\in\mathcal{C}_{N-1}[h(I)]$, where $\mathcal{C}_0[I]$ denote the class of positive functions on the interval I. From the condition $h(x)\in\mathcal{K}_N[I]$ which means $(-1)^ih^{(i+1)}\ge0$ for $0\le i\le N-1$, $\mathcal{K}_N[I]\subset\mathcal{K}_{N-1}[I]$ which can be deduced readily from (4) and (15), and [22, Theorem A] which states that if $h\in\mathcal{K}_N[I]$ and $f\in\mathcal{C}_N[h(I)]$ then $f(h)\in\mathcal{C}_N[I]$ for $N\in\mathbb{N}\cup\{0,\infty\}$, it is easy to see that $-f'(h)/f(h)\in\mathcal{C}_{N-1}[I]$, that is, $(-1)^i[-f'(h)/f(h)]^{(i)}\ge0$ for $0\le i\le N-1$. Therefore, directly calculating gives

$$\begin{aligned}
(-1)^k\bigl[\ln f\bigl(h(x)\bigr)\bigr]^{(k)}&=(-1)^k\left[\frac{f'\bigl(h(x)\bigr)}{f\bigl(h(x)\bigr)}h'(x)\right]^{(k-1)}\\
&=(-1)^k\sum_{i=0}^{k-1}\binom{k-1}{i}\left[\frac{f'\bigl(h(x)\bigr)}{f\bigl(h(x)\bigr)}\right]^{(i)}h^{(k-i)}(x)\\
&=\sum_{i=0}^{k-1}\binom{k-1}{i}\left\{(-1)^i\left[-\frac{f'\bigl(h(x)\bigr)}{f\bigl(h(x)\bigr)}\right]^{(i)}\right\}\left[(-1)^{k-i-1}h^{(k-i)}(x)\right] \qquad (27)\\
&\ge0
\end{aligned}$$

for $0 \le k \le N$. The proof is complete. $\square$

Proof of Theorem 2. By standard arguments, it follows that

$$(-1)^k\left[\ln\prod_{i=1}^n\bigl(f_i(x)\bigr)^{\alpha_i}\right]^{(k)}=\sum_{i=1}^n\alpha_i\bigl\{(-1)^k[\ln f_i(x)]^{(k)}\bigr\}\ge0 \tag{28}$$

for $1 \le k \le N$, since $f_i(x) \in \mathcal{L}_N[I]$, that is, $(-1)^k[\ln f_i(x)]^{(k)} \ge 0$ hold for $1 \le k \le N$ and $1 \le i \le n$, and $\alpha_i \ge 0$ for $1 \le i \le n$. The proof is complete. $\square$

Proof of Theorem 3. From $f(x) \in \mathcal{L}_N[I]$, it follows that $(-1)^k[\ln f(x)]^{(k)} \ge 0$ for $1 \le k \le N$. This is equivalent to $[\ln f(x)]^{(2i)} \ge 0$ for $1 \le 2i \le N$ and $[\ln f(x)]^{(2i-1)} \le 0$ for $1 \le 2i-1 \le N$, and then $[\ln f(x)]^{(2i)}$ is decreasing for $1 \le 2i \le N-1$ and $[\ln f(x)]^{(2i-1)}$ is increasing for $1 \le 2i-1 \le N-1$. Therefore, from $\alpha > 0$ it follows that $\{\ln[f(x)/f(x+\alpha)]\}^{(2i)} = [\ln f(x)]^{(2i)} - [\ln f(x+\alpha)]^{(2i)} \ge 0$ for $1 \le 2i \le N-1$ and $\{\ln[f(x)/f(x+\alpha)]\}^{(2i-1)} \le 0$ for $1 \le 2i-1 \le N-1$, that is, $(-1)^i\{\ln[f(x)/f(x+\alpha)]\}^{(i)} \ge 0$ for $1 \le i \le N-1$. The proof is complete. $\square$

Proof of Theorem 4. Taking the logarithm of $G(x)$, using the first Binet's formula for $\ln\Gamma(x)$

$$\ln\Gamma(x+1)=\left(x+\frac12\right)\ln x-x+\frac{\ln(2\pi)}2+\int_0^\infty\left[\frac12-\frac1t+\frac1{e^t-1}\right]\frac{e^{-xt}}t\,\mathrm{d}\,t \tag{29}$$

which can be found in [24] and [25, p. 106], and differentiating successively gives

$$(-1)^\ell[\ln G(x)]^{(\ell)}=\int_0^\infty t^{\ell-1}\left[\frac12-\frac1t+\frac1{e^t-1}\right]\left[\sum_{i=1}^n a_i^\ell e^{-a_ixt}-s^\ell e^{-sxt}\right]\mathrm{d}\,t \tag{30}$$

for any nonnegative integer ℓ.

Since the derivative $\delta'(t)$ of the function

$$\delta(t)=\frac12-\frac1t+\frac1{e^t-1} \tag{31}$$

is decreasing and positive in $(0,\infty)$, see [13], thus it is easy to obtain that $\delta(t)$ is increasing and positive in $(0,\infty)$, see also [5]. Therefore, it is sufficient to prove

$$\sum_{i=1}^n a_i^k e^{-a_iu}\ge s^ke^{-su} \tag{32}$$

for all $u = xt \ge 0$, which is equivalent to

$$\sum_{i=1}^n a_i^k\exp\left[\left(\sum_{j\ne i}a_j\right)u\right]\ge s^k. \tag{33}$$

It is obvious that inequality (33) holds if

$$\sum_{i=1}^{n} a_i^k \geq s^k = \left(\sum_{i=1}^{n} a_i\right)^k, \tag{34}$$

which can be rewritten as

$$\sum_{i=1}^{n} \left(\frac{a_i}{\sum_{j=1}^{n} a_j}\right)^k \geq 1. \tag{35}$$

Since $a_i/\sum_{j=1}^{n} a_j < 1$, then for all $1 \leq p < k$

$$\sum_{i=1}^{n} \left(\frac{a_i}{\sum_{j=1}^{n} a_j}\right)^p > \sum_{i=1}^{n} \left(\frac{a_i}{\sum_{j=1}^{n} a_j}\right)^k \geq 1. \tag{36}$$

This implies $(-1)^q[\ln G(x)]^{(q)} \geq 0$ for all $1 \leq q \leq k$. The proof is complete. □

Proof of Theorem 5. It is well known [1, 24, 25] that for $x > 0$ and $r > 0$

$$\frac{1}{x^r} = \frac{1}{\Gamma(r)} \int_0^\infty t^{r-1} e^{-xt} \,\mathrm{d}t. \tag{37}$$

The psi and polygamma functions can be expressed [1, 24, 25] as

$$\psi(x) = -\gamma + \int_0^\infty \frac{e^{-t} - e^{-xt}}{1 - e^{-t}} \,\mathrm{d}t \tag{38}$$

and

$$\psi^{(k)}(x) = (-1)^{k+1} \int_0^\infty \frac{t^k e^{-xt}}{1 - e^{-t}} \,\mathrm{d}t, \quad k \in \mathbb{N}. \tag{39}$$

Taking the logarithm of $F(x)$, differentiating with respect to x, utilizing formulas (37), (38) and (39), and simplifying yields

$$\begin{aligned} [\ln F(x)]^{(k)} &= m\left[\psi^{(k-1)}(x+b) - m^{k-1}\psi^{(k-1)}(mx+a) + \frac{(-1)^k(k-1)!}{2x^k}\right] \\ &= (-1)^k m \int_0^\infty t^{k-1}\left(\frac{1}{2} - \frac{m^{k-1}e^{-[(m-1)x+a]t} - e^{-bt}}{1-e^{-1}}\right)e^{-xt} \,\mathrm{d}t \end{aligned} \tag{40}$$

for $k \in \mathbb{N}$.

In order that $(-1)^k[\ln F(x)]^{(k)} \geq 0$, it is sufficient to show

$$\frac{1}{2} - \frac{m^{k-1}e^{-[(m-1)x+a]t} - e^{-bt}}{1-e^{-1}} \geq 0 \tag{41}$$

for all $t \ge 0$, which is equivalent to

$$\begin{aligned} x &\ge \frac{1}{m-1}\left[(k-1)\ln m - \ln\frac{1-e^{-t}+2e^{-bt}}{2e^{-at}}\right] \\ &= \frac{1}{m-1}\left[(k-1)\ln m + \ln 2 - \ln\frac{1-u+2u^b}{u^a}\right] \\ &= \frac{(k-1)\ln m + \ln 2 + a\ln u - \ln\left(1-u+2u^b\right)}{m-1} \end{aligned} \tag{42}$$

for all $t \ge 0$ and $0 < u = e^{-t} \le 1$. The first conclusion follows.

If $b = 1$ and $a > 1/2$, the function $a\ln u - \ln(1+u) \le -\ln 2$ is increasing in $(0,1]$; if $b = 1$ and $0 < a \le 1/2$, the function $a\ln u - \ln(1+u)$ for $u \in (0,1]$ has a maximum $\ln\left[a^a(1-a)^{1-a}\right]$. By calculus, it is easy to show that the function $\ln\left[x^x(1-x)^{1-x}\right]$ is decreasing in $x \in (0,1/2]$, and then $0 > \ln\left[a^a(1-a)^{1-a}\right] \ge -\ln 2$ for $0 < a \le 1/2$. This implies the second conclusion (22).

If $b \ne 1$, then inequality (42) is valid if

$$x \ge \frac{(k-1)\ln m + \ln 2 - \ln\left(1-u+2u^b\right)}{m-1} \tag{43}$$

for $u \in (0,1]$. It is easy to obtain that the function $2u^b - u$ has a unique critical point which is a minimum point $(2b)^{1/(1-b)}$ in $(0,1]$ if $b > 1$, has an unique critical point which is a maximum point $(2b)^{1/(1-b)}$ in $(0,1]$ if $0 < b \le 1/2$, and is increasing in $(0,1]$ if $1 > b > 1/2$. Therefore, if $0 < b < 1$ then $\ln\left(1-u+2u^b\right) > 0$, if $b > 1$ then $\ln\left(1-u+2u^b\right) \ge \ln\left[1+2(2b)^{b/(1-b)} - (2b)^{1/(1-b)}\right]$ in $(0,1]$. This means that $(-1)^k[\ln F(x)]^{(k)} \ge 0$ holds if

$$x \begin{cases} > \dfrac{(k-1)\ln m + \ln 2}{m-1} & \text{for } 0 < b < 1, \\ \ge \dfrac{(k-1)\ln m + \ln 2 - \ln\left[1+(1-b)\left(2b^b\right)^{1/(1-b)}\right]}{m-1} & \text{for } b > 1. \end{cases} \tag{44}$$

The proof is complete. □

Proof of Theorem 6. Straightforward computation yields

$$\begin{aligned} [\ln H(x)]' &= (\ln q)\sum_{i=1}^{\infty}\left[\sum_{j=1}^{n}\frac{a_j q^{i+a_j x}}{1-q^{i+a_j x}} - \frac{s q^{i+sx}}{1-q^{i+sx}}\right] \\ &= (\ln q)\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\left[\sum_{k=1}^{n} a_k q^{(j+a_k x)i} - s q^{(j+sx)i}\right] \end{aligned} \tag{45}$$

and

$$[\ln H(x)]^{(\ell)} = (\ln q)^\ell \sum_{i=1}^{\infty} i^{\ell-1} \sum_{j=1}^{\infty} \left[\sum_{k=1}^{n} a_k^\ell q^{(j+a_k x)i} - s^\ell q^{(j+sx)i} \right] \tag{46}$$

for $\ell \in \mathbb{N}$. Thus it suffices to show that

$$\sum_{k=1}^{n} a_k^\ell q^{(j+a_k x)i} \ge s^\ell q^{(j+sx)i} \tag{47}$$

which is equivalent to

$$\sum_{k=1}^{n} a_k^\ell q^{a_k ix} \ge s^\ell q^{six}. \tag{48}$$

Furthermore, it is clear that inequality (48) is equivalent to (32), which has already been established in the proof of Theorem 4. The proof is complete. □

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