# COMPARISONS OF TWO INTEGRAL INEQUALITIES WITH HERMITE-HADAMARD-JENSEN'S INTEGRAL INEQUALITY

FENG QI AND MENG-LONG YANG

Abstract. Certain comparisons of Iyengar-Mahajani's and Kesava Menon's integral inequalities with Hermite-Hadamard-Jensen's integral inequalities are considered and some mistakes in the paper [*On certain inequalities by Iyengar and Kesava Menon*, Octogon Math. Mag. **4** (1996), no. 1, 9–11.] are corrected. Some applications of these inequalities to elementary functions are carried out and several inequalities involving mean values are obtained.

## 1. Introduction

In [4, 6], motivated by Problem 121 in [7, p. 62] (See also [8]), K. S. K. Iyengar proposed proving the following inequality by means of geometrical considerations.

**Theorem 1.** *Let $f(x)$ be continuous and not identically a constant on $[a,b]$. If $M$ is the upper bound of $|f'(x)|$ in $(a,b)$, then*

$$\frac{1}{4M}[f(b)-f(a)]^2-\frac{M(b-a)^2}{4}<\int_a^b f(t)\,\mathrm{d}t-\frac{1}{2}(b-a)[f(a)+f(b)] < \frac{M(b-a)^2}{4}-\frac{1}{4M}[f(b)-f(a)]^2. \quad (1)$$

*Inequality* (1) *is sharp in the sense that it can not be improved.*

In [6], among other things, an alternative geometrical proof of inequality (1) was provided by G. S. Mahajani. So, we would like to call inequality (1) Iyengar-Mahajani's integral inequality.

In [5], the following integral inequalities are established by analytic approach.

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**Theorem 2.** *If $f'(b)$ is not equal to zero, then*

$$\int_a^b f(x)\,\mathrm{d}x \gtreqless (b-a)f(a) + \frac{1}{2}\frac{[f(b)-f(a)]^2}{f'(b)} \tag{2}$$

*according as $f''(x) \gtreqless 0$ in $(a,b)$.*

Inequalities in (2) are called Kevasa Menon's integral inequalities.

The following integral inequality about convex functions is well known.

**Theorem 3.** *For $f$ being convex on $[a,b]$,*

$$(b-a)f\left(\frac{a+b}{2}\right) < \int_a^b f(x)\,\mathrm{d}x < (b-a)\left[\frac{f(a)+f(b)}{2}\right]. \tag{3}$$

Inequality (3) is often called Hadamard's inequality [1], Hermite-Hadamard's inequality [10], or Jensen-Hadamard's inequality [11], since ones are not aware that this inequality appeared first in a letter by Hermite on November 22, 1881 to the journal *Mathesis* and an extract from that letter was published in *Mathesis* **3** (1883), page 82. For a historical consideration of (3), please refer to [3, Chapter 1] and the reference therein. Here, we would like to call inequality (3) Hermite-Hadamard-Jensen's integral inequality.

Some comparisons between inequalities (1), (2) and (3) had been studied in [11]. However, because the inequality for the case of $f''(x) > 0$ in (2) was reversedly referenced, all of results related to (2) in [11] must be wrong. For example, the inequality (12) in [11], $e^{2b} - e^{2a} < 2(b-a)e^{a+b}$, does not hold for $a=0$ and $b=1$.

In this short note, we shall reconsider certain comparisons of Iyengar-Mahajani's integral inequality and Kesava Menon's integral inequality in the case of $f''(x) > 0$ with Hermite-Hadamard-Jensen's integral inequality and then correct some errors in [11].

Our main results are as follows.

**Theorem 4.** *Denote*

$$S_0(a,b) = \frac{f(b)-f(a)}{b-a}. \tag{4}$$

(1) *Let $f'(x)$ is increasing and $|f'(x)| \le M$ on $[a,b]$.*
   (a) *The right hand side of inequality (3) is better than the second inequality in (1).*

(b) *If*

$$f'(a)f'(b) > [S_0(a,b)]^2, \tag{5}$$

*then the left hand side of inequality* (3) *is better than the first inequality in* (1).

(2) *Let* $f'(b) \neq 0$ *and* $f''(x) > 0$ *on* $[a,b]$.

(a) *If*

$$\frac{[S_0(a,b)]^2}{f'(b)} < f'(a), \tag{6}$$

*then the left hand side of* (3) *is better than inequality* (2).

(b) *If*

$$\frac{[S_0(a,b)]^2}{f'(b)} > f'\left(\frac{a+b}{2}\right), \tag{7}$$

*then the left hand side of* (3) *is weaker than inequality* (2).

Moreover, applications of these theorems to elementary functions are carried out and several inequalities involving mean values are obtained.

## 2. Proofs of main results

**2.1. Comparison between (1) and (3).** In this subsection, assume that $f'(x)$ is increasing and $|f'(x)| \le M$ on $[a,b]$. It is easy to see that $M = f'(b)$ or $M = -f'(a)$.

It is clear that the right hand side of inequality (3) is better than the second inequality in (1), since $\frac{M(b-a)^2}{4} - \frac{1}{4M}[f(b)-f(a)]^2$ is positive.

The first inequality in (1) is better (weaker) than the left hand side of (3) if and only if

$$\frac{f(a)+f(b)}{2} \gtrless f\left(\frac{a+b}{2}\right) + \frac{(b-a)M}{4} - \frac{b-a}{4M}\left[\frac{f(b)-f(a)}{b-a}\right]^2. \tag{8}$$

By Lagrange's mean value theorem on $\left[a, \frac{a+b}{2}\right]$ and $\left[\frac{a+b}{2}, b\right]$ respectively, we obtain

$$\frac{f(a)+f(b)}{2} = f\left(\frac{a+b}{2}\right) + \frac{b-a}{4}[f'(\xi_2) - f'(\xi_1)], \tag{9}$$

where $\xi \in \left(a, \frac{a+b}{2}\right)$ and $\xi_2 \in \left(\frac{a+b}{2}, b\right)$. So, inequality (8) is satisfied if

$$f'(x) - f'(y) \gtrless M - \frac{1}{M}\left[\frac{f(b)-f(a)}{b-a}\right]^2 \tag{10}$$

for all $y \in \left(a, \frac{a+b}{2}\right)$ and $x \in \left(\frac{a+b}{2}, b\right)$. Therefore, if

$$f'(b) - f'(a) < M - \frac{1}{M}\left[\frac{f(b)-f(a)}{b-a}\right]^2 \tag{11}$$

holds, then the first inequality in (1) is weaker than the left hand side of (3).

If $M = f'(b)$, then inequality (11) is reduced to (5). If $M = -f'(a)$, then inequality (11) is also reduced to inequality (5).

Now it is seen that inequality (5) is a sufficient condition such that the left hand side of inequality (3) is better than the first inequality in (1).

2.2. **Comparison between** (2) **and** (3)**.** In this subsection, let us suppose that $f'(b) \neq 0$ and $f''(x) > 0$ on $[a, b]$.

It is clear that the first derivative $f'(x)$ is strictly increasing on $[a, b]$.

The left hand side of (3) is better (weaker) than inequality (2) if and only if

$$(b-a)f(a) + \frac{1}{2}\frac{[f(b)-f(a)]^2}{f'(b)} \leqslant (b-a)f\left(\frac{a+b}{2}\right), \tag{12}$$

which are equivalent with

$$\frac{1}{2}\frac{[f(b)-f(a)]^2}{f'(b)} \leqslant (b-a)\left[f\left(\frac{a+b}{2}\right) - f(a)\right] = \frac{(b-a)^2}{2}f'(\xi), \tag{13}$$

where $\xi \in \left(a, \frac{a+b}{2}\right)$. Therefore, inequality (12) holds true if for $x \in \left(a, \frac{a+b}{2}\right)$

$$\left[\frac{f(b)-f(a)}{b-a}\right]^2 \frac{1}{f'(b)} \leqslant f'(x). \tag{14}$$

Hence, if inequality (6) holds, then the left hand side of (3) is better than inequality (2); if inequality (6) holds, then the left hand side of (3) is weaker than inequality (2).

## 3. Applications of inequalities (1), (2) and (3)

3.1. Let $f(x) = \pm \ln x$ in (2). It is obvious that $f''(x) = \mp \frac{1}{x^2}$ for $0 < a \leq x \leq b$. Then a simple computation gives

$$\frac{1}{b-a}\int_a^b \ln x \,\mathrm{d}x < \ln a + \frac{b(b-a)}{2}\left[\frac{\ln b - \ln a}{b-a}\right]^2. \tag{15}$$

Recall that the so-called identric or exponential mean and logarithmic mean of two positive numbers $a$ and $b$ are defined (see [2, 9]) respectively by

$$I(a,b) = \exp\left(\frac{1}{b-a}\int_a^b \ln x \,\mathrm{d}x\right), \tag{16}$$

$$L(a,b) = \frac{b-a}{\ln b - \ln a}. \tag{17}$$

By this notation it is easy to see that (15) transforms into

$$\ln\frac{I(a,b)}{a}<\frac{b(b-a)}{2[L(a,b)]^2}. \tag{18}$$

3.2. For $0<a<b$, applying $f(x)=-\ln x$ to the first inequality in (1) and the right hand side of (3) and remarking that $M=\frac{1}{a}$. Straightforward computation yields

$$\frac{\ln a+\ln b}{2}<\frac{1}{b-a}\int_a^b\ln x\,\mathrm{d}x<\frac{\ln a+\ln b}{2}+\frac{b-a}{4a}-\frac{a(b-a)}{4}\left[\frac{\ln b-\ln a}{b-a}\right]^2, \tag{19}$$

which can be rewritten as

$$0<\ln\frac{I(a,b)}{G(a,b)}<\frac{b-a}{4a}\left\{1-\frac{a^2}{[L(a,b)]^2}\right\}, \tag{20}$$

where $G(a,b)=\sqrt{ab}$ denotes the geometric mean of two positive numbers $a$ and $b$.

3.3. Set $f(x)=e^x$ in (2), the first inequality of (1) and the right hand side of (3).

After certain elementary calculus, inequality (2) gives us

$$e^{2b}-e^{2a}>2(b-a)e^{a+b}, \tag{21}$$

which can be rewritten as

$$\frac{e^b-e^a}{b-a}>e^{(a+b)/2}. \tag{22}$$

This inequality is equivalent to applying the left hand side of (3) to the exponential function $e^x$. So inequality (2) and the left hand side of (3) is equivalent each other when $f(x)=e^x$.

From the first inequality in (1) and the right hand side of (3) one obtains

$$\frac{e^a+e^b}{2}-\frac{e^2(b-a)}{4}<\frac{e^b-e^a}{b-a}\,\frac{3e^b+e^a}{4e^b} \tag{23}$$

and

$$\frac{e^b-e^a}{b-a}<\frac{e^a+e^b}{2}. \tag{24}$$

Lagrange's mean value theorem applied to the function $e^{2x}$ gives

$$e^{2b}-e^{2a}=2(b-a)e^{2\xi}>2(b-a)e^{2a} \tag{25}$$

which is much weaker than (21).

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(F. Qi) Research Institute of Mathematical Inequality Theory, Henan Polytechnic University, Jiaozuo City, Henan Province, 454010, China

*E-mail address*: `qifeng@hpu.edu.cn, fengqi618@member.ams.org`

*URL*: `http://rgmia.vu.edu.au/qi.html`

(M.-L. Yang) Department of Computer, Jiaozuo Teachers College, Jiaozuo City, Henan Province, 454001, China