# SOME IDENTITIES FOR MEANS AND APPLICATIONS

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ABSTRACT. In this paper, the power-exponential mean is introduced, several identities involving exponential mean and power-exponential mean are given. As applications, some new inequalities for means are presented.

## 1. Introduction

Exponential mean or identical mean of two unequal positive numbers $a$ and $b$ is defined by

$$E = E(a,b) = \begin{cases} e^{-1}\left(\dfrac{a^a}{b^b}\right)^{\frac{1}{a-b}}, & a \neq b; \\ a, & a = b. \end{cases} \tag{1.1}$$

Regarding the exponential mean $E(a,b)$ there are many interesting and useful results, such as (see [5, 9, 10])

$$G(a,b) < L(a,b) < \frac{A(a,b)+G(a,b)}{2} < E(a,b) < A(a,b), \tag{1.2}$$

where

$$A = A(a,b) = \frac{a+b}{2}; \quad G = G(a,b) = \sqrt{ab}; \quad L = L(a,b) = \begin{cases} \frac{b-a}{\ln b - \ln a}, & a \neq b; \\ a, & a = b. \end{cases},$$

and

$$E(a,b) \;>\; A_{\frac{2}{3}}(a,b), \tag{1.3}$$

$$L(a,b) + E(a,b) \;<\; A(a,b) + G(a,b), \tag{1.4}$$

where $A_t = A_t(a,b) = \left(\frac{a^t+b^t}{2}\right)^{\frac{1}{t}}$, etc. .

In [10], Zhen-hang Yang considered two-parameter mean related $E(a,b)$ which is defined by

$$\mathcal{H}_E(a,b;p,q) = \begin{cases} \left(\dfrac{E(a^p,b^p)}{E(a^q,b^q)}\right)^{\frac{1}{p-q}}, & p \neq q; \\ G_{E,p}(a,b), & p = q \neq 0; \\ G(a,b), & p = q = 0. \end{cases}, \tag{1.5}$$

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*Date*: August 18, 2005.

2000 *Mathematics Subject Classification.* Primary 26E60, 26D20.

*Key words and phrases.* Mean identity, Exponential mean, Power-exponential mean, Compound mean, inequality.

where

$$G_{E,p}(a,b) = Y_p(a,b) = Y^{\frac{1}{p}}(a^p, b^p) = Y_p,$$
$$Y(a,b) = Ee^{1-\frac{G^2}{L^2}}.$$

It was proved that (see [10, 11])

**Theorem 1.** *1) $\mathcal{H}_E(p,q)$ is strictly increasing in $p$ or $q$ on $(-\infty,+\infty)$.*

*2) $\mathcal{H}_E(p,q)$ are strictly log-concave with respect to either $p$ or $q$ on $(0,+\infty)$, and log-convex on $(-\infty,0)$.*

*3) $\mathcal{H}_E(p,1-p)$ are strictly increasing in $p$ on $(-\infty,\frac{1}{2})$, and strictly decreasing on $(\frac{1}{2},+\infty)$.*

*4) If $p+q>0$ with $p\neq q$, then*

$$G_{E,\frac{p+q}{2}} > \mathcal{H}_E(p,q) > \sqrt{G_{E,p}G_{E,q}}. \tag{1.6}$$

*Inequality (1.6) is reversed if $p+q<0$ with $p\neq q$.*

In [6, 7, 8], J. Sandor and Wan-ran Wang investigated identity involving the identical mean, logarithmic mean, Stolarsky mean and power mean, and presented the following results:

$$\ln\frac{E^2(\sqrt{a},\sqrt{b})}{E(a,b)} = \frac{G-L}{L} \tag{1.7}$$

$$\ln\frac{E^3(\sqrt[3]{a},\sqrt[3]{b})}{E(a,b)} = 2\left(\frac{A_{\frac{1}{3}}^{\frac{1}{3}}G^{\frac{2}{3}}}{L}-1\right) \tag{1.8}$$

$$\ln\frac{E^t(a^{\frac{1}{t}},b^{\frac{1}{t}})}{E(a,b)} = (t-1)\left[\frac{S_{t-2}^{t-2}(a^{\frac{1}{t}},b^{\frac{1}{t}})G^{\frac{2}{t}}(a,b)-L(a,b)}{L(a,b)}-1\right], \tag{1.9}$$

where $S_p(a,b) = \left(\frac{b^p-a^p}{p(b-a)}\right)^{\frac{1}{p-1}}$ $(p\neq 1)$, $S_0(a,b)=L(a,b)$, $S_1(a,b)=E(a,b)$.

Applying the above identities, they obtained some new inequalities.

The purpose of this paper is to give other general identities and inequalities concerning exponential mean and power-exponential mean, and corresponding inequalities will be presented. In section 2, the power-exponential mean and its meanings are introduced; In section 3, certain identities for exponential mean are stated; In section 4, we will present corresponding inequalities.

## 2. Power-exponential Mean Z(a,b)

2.1. **Definition and Property.** Let us consider weighted geometric mean of unequal positive numbers $a$ and $b$: $G(a,b;p,q)=a^qb^p$, where $p,q>0$ with $p+q=1$.Setting $p=\frac{a}{a+b}, q=\frac{b}{a+b}$, obviously $a^{\frac{a}{a+b}}b^{\frac{b}{a+b}}$ is also a mean of $a$ and $b$, which is called power-exponential mean and denote by $Z(a,b)$.

It is easy to obtain the properties of $Z(a,b)$.

**Property 1** $Z(a,b)$ is symmetric with respect to $a$ and $b$, i.e.

$$Z(a,b)=Z(b,a).$$

**Property 2** $Z(a,b)$ is homogeneous with respect to $a$ and $b$, i.e.

$$Z(ta,tb) = tZ(a,b) \text{ for } t > 0.$$

**Property 3** $Z(a,b)$ has an upper bound and lower bound, i.e.

$$\min(a,b) \le Z(a,b) \le \max(a,b).$$

2.2. **Two Other Meanings of $Z(a,b)$.** The so-called Gini mean of positive numbers $a$ and $b$ is defined by

$$\begin{aligned} G_{s,t}(a,b) &= \left(\frac{a^s+b^s}{at+b^t}\right)^{\frac{1}{s-t}} \quad (s \neq t), \\ G_{t,t}(a,b) &= \lim_{s\to t}\left(\frac{a^s+b^s}{at+b^t}\right)^{\frac{1}{s-t}} = Z^{\frac{1}{t}}(a^t,b^t) \quad (t\neq 0), \\ G_{0,0}(a,b) &= \lim_{t\to 0} Z^{\frac{1}{t}}(a^t,b^t) = G(a,b) \quad (t=0). \end{aligned}$$

It shows that $Z(a,b)$ is a case of limit for Gini mean.

Let

$$Z_t(a,b) = \begin{cases} Z^{\frac{1}{t}}(a^t,b^t), & t \neq 0; \\ G(a,b), & t = 0. \end{cases}$$

Then according to the monotonicity and log-convexity of Gini mean, we have

**Theorem 2.** [10, Corollary 2.1] *$Z_t(a,b)$ is increasing in $t$ on interval $(-\infty,+\infty)$.*

**Theorem 3.** [11, 1) of Conclusion 1] *.$Z_t(a,b)$ is strictly log-convex in $t$ on interval $(-\infty,0)$, and log-concave on interval $(0,+\infty)$.*

In addition, $Z(a,b)$ has another concise expression.

**Theorem 4.** [10, Remark 4.1]

$$Z(a,b) = \frac{E(a^2,b^2)}{E(a,b)}.$$

2.3. **Some Inequalities for Power-exponential Mean.** Concerning $Z(a,b)$ with $a \neq b$, there are the following inequalities (See [10, eq. 4.3, 4.5]:

$$\sqrt{ab} < \frac{a+b}{2} < \left(\frac{a+b}{\sqrt{a}+\sqrt{b}}\right)^2 < Z(a,b) < \frac{a^2+b^2}{a+b}, \tag{2.1}$$

$$\sqrt{ab} < E(a,b) < Z^2\left(\sqrt{a},\sqrt{b}\right) < E\exp(1-\frac{G^2}{L^2}) < Z(a,b). \tag{2.2}$$

The following inequalities were presented by [11, eq. 3.9, 3.10]:

$$\begin{aligned} G^{\frac{2}{3}}A_2^{\frac{2}{3}}A^{-\frac{1}{3}} &< G^{\frac{1}{2}}A_{\frac{3}{2}}^{\frac{3}{4}}A_{\frac{1}{2}}^{-\frac{1}{4}} < G^{\frac{2}{5}}A_{\frac{4}{3}}^{\frac{4}{5}}A_{\frac{1}{3}}^{-\frac{1}{5}} < A < A_{\frac{4}{5}}^{\frac{4}{3}}A_{\frac{1}{5}}^{-\frac{1}{3}} \\ &< A_{\frac{3}{4}}^{\frac{3}{2}}A_{\frac{1}{4}}^{-\frac{1}{2}} < A_{\frac{2}{3}}^{2}A_{\frac{1}{3}}^{-1} < A_{\frac{3}{5}}^{3}A_{\frac{2}{5}}^{-2} < Z_{\frac{1}{2}}, \end{aligned} \tag{2.3}$$

where $A_p = (\frac{a^p+b^p}{2})^{\frac{1}{p}}$, $Z_p = Z^{\frac{1}{p}}(a^p,b^p), Z(a,b) = a^{\frac{a}{a+b}}b^{\frac{b}{a+b}}$.

$$G^{\frac{2}{3}}Z^{\frac{1}{3}} < G^{\frac{1}{2}}E_{\frac{3}{2}}^{\frac{3}{4}}E_{\frac{1}{2}}^{-\frac{1}{4}} < G^{\frac{2}{5}}Z_{\frac{1}{3}}^{\frac{1}{5}}Z_{\frac{2}{3}}^{\frac{2}{5}} < E < Z_{\frac{1}{5}}^{\frac{1}{3}}Z_{\frac{2}{5}}^{\frac{2}{3}} \tag{2.4}$$
$$< E_{\frac{3}{4}}^{\frac{3}{2}}E_{\frac{1}{4}}^{-\frac{1}{2}} < Z_{\frac{1}{3}} < E_{\frac{3}{5}}^{3}E_{\frac{2}{5}}^{-2} < Y_{\frac{1}{2}},$$

where $Z_p = Z^{\frac{1}{p}}(a^p, b^p)$, $E_p = E^{\frac{1}{p}}(a^p, b^p)$, $Y_p = Y^{\frac{1}{p}}(a^p, b^p)$, $Y(a,b) = Ee^{1-\frac{G^2}{L^2}}$.

From (2.3) and (2.4), we can obtain

$$A < A_{\frac{3}{4}}^{\frac{3}{2}}A_{\frac{1}{4}}^{-\frac{1}{2}} < Z_{\frac{1}{2}}, \tag{2.5}$$

$$E < Z_{\frac{1}{3}} < Y_{\frac{1}{2}}, \tag{2.6}$$

respectively, which may be transformed into

$$A_2 < A_{\frac{3}{2}}^{\frac{3}{2}}A_{\frac{1}{2}}^{-\frac{1}{2}} < Z, \tag{2.7}$$

$$E_3 < Z < Y_{\frac{3}{2}}. \tag{2.8}$$

That $A_{\frac{3}{2}}^{\frac{3}{2}}A_{\frac{1}{2}}^{-\frac{1}{2}} < Z$ may be transformed into $Z > \dfrac{a^{\frac{3}{2}}+b^{\frac{3}{2}}}{a^{\frac{1}{2}}+b^{\frac{1}{2}}} = a + b - \sqrt{ab}$, i.e.

$$\frac{Z+G}{2} > A. \tag{2.9}$$

Let us consider the right approximations of $Z$ in $A_p$. By (2.7) we have

$$Z > \sqrt{\frac{a^2+b^2}{2}} = A_2. \tag{2.10}$$

Here $p = 2$ is the best constant? The following Theorem answer the question:

**Theorem 5.** *For positive numbers $a$ and $b$, the following is always true:*

$$Z(a,b) > \left(\frac{a^p+b^p}{2}\right)^{\frac{1}{p}}, \tag{2.11}$$

*where $p = 2$ is the best constant.*

*Proof.* Set $x = \frac{b}{a}$, then inequality (2.11) is equivalent to

$$x^{\frac{x}{x+1}} > \left(\frac{x^p+1}{2}\right)^{\frac{1}{p}}. \tag{2.12}$$

Let $f(x) = \ln x^{\frac{x}{x+1}} - \ln\left(\frac{x^p+1}{2}\right)^{\frac{1}{p}} = \frac{x}{x+1}\ln x - \frac{1}{p}\ln\left(\frac{x^p+1}{2}\right)$.Then

$$\begin{aligned} f'(x) &= \frac{x+1+\ln x}{(x+1)^2} - \frac{x^{p-1}}{x^p+1} \\ &= \frac{1}{(x+1)^2}\left(\ln x - \frac{(x+1)(x^{p-1}-1)}{x^p+1}\right) \\ &= \frac{1}{(x+1)^2}\left(\ln x - \frac{x^p+x^{p-1}-x-1}{x^p+1}\right). \end{aligned}$$

And let

$$g(x) = \ln x - \frac{x^p+x^{p-1}-x-1}{x^p+1} \tag{2.13}$$

further. Then $g(1) = 0$ and

$$\begin{aligned} g'(x) &= \frac{1}{x} - \frac{[px^{p-1}+(p-1)x^{p-2}-1](x^p+1)-(x^p+x^{p-1}-x-1)px^{p-1}}{(x^p+1)^2} \\ &= \frac{(x+1)^2}{x(x^p+1)^2}\left[\frac{x^{2p-1}+1}{x+1} - (p-1)x^{p-1}\right]. \end{aligned}$$

Obviously, $g'(1) = 2 - p$.

1) If $g'(1) = 2 - p \geqslant 0$. By inequality (2.1), $p > 1$, we have

$$\frac{x^{2p-1}+1}{x+1} - x^{p-1} = \frac{(x^p-1)(x^{p-1}-1)}{x+1} > 0,$$

therefor

$$\frac{x^{2p-1}+1}{x+1} - (p-1)x^{p-1} > x^{p-1} - (p-1)x^{p-1} = (2-p)x^{p-1} > 0,$$

i.e. $g'(x) > 0$. So $g(x) > g(1) = 0$ if $x > 1$, which shows $f'(x) > 0$, consequently we have $f(x) > f(1) = 0$; Likewise we have $f(x) > f(1) = 0$ if $0 < x < 1$. Thus inequality (2.12) is always valid if $x > 0$ with $x \neq 1$, i.e. (2.11) holds.

2) If $g'(1) = 2 - p < 0$. Because $g'(+\infty) = 1 > 0$ and $g'(x)$ is continuous on $(0, +\infty)$, by the properties of continuous functions, there exists $x_1 \in (1, +\infty)$, such that $g'(x_1) = 0$. If $1 < x < x_1$, then $g'(x) < 0$; While $x_1 < x < +\infty$ then $g'(x) > 0$. Thus $g(x) < g(1) = 0$ if $1 < x < x_1$, and then $f'(x) < 0$, thereby $f(x) < f(1) = 0$.

On the other hand, since $f\left(\frac{1}{x}\right) = f(x)$, we have

$$f(+\infty) = f(+0) = \lim_{x \to +0}\left[\frac{x}{x+1}\ln x - \frac{1}{p}\ln\frac{x^p+1}{2}\right] = \frac{1}{p}\ln 2 > 0$$

It is obvious that $f(x)$ does not have certain sign on $(0, +\infty)$, i.e. inequality (2.12) does not always hold, naturally inequality (2.11) is not valid.

Combining 1) with 2), this complete the proof. ∎

## 3. Some Expressions of E(a,b)

**Theorem 6.** *Let $p, q \in \mathbb{R}$ with $p + q = 1$. For positive numbers $a$ and $b$ ,we have*

$$E(a,b) = a^p b^q \exp\left[\frac{qa+pb}{L(a,b)} - 1\right]. \tag{3.1}$$

*Proof.*

$$\begin{aligned} \ln E(a,b) &= \frac{b\ln b - a\ln a}{b-a} - 1 \\ &= \frac{b\ln b - b\ln a + b\ln a - a\ln a}{b-a} - 1 \\ &= \frac{b\ln b - b\ln a + b\ln a - a\ln a}{b-a} - 1 \\ &= \frac{b}{L(a,b)} + \ln a - 1, \end{aligned}$$

i.e.

$$E(a,b)=ae^{\frac{b}{L(a,b)}-1}. \tag{3.2}$$

In this way, we have

$$E(a,b)=be^{\frac{a}{L(a,b)}-1}. \tag{3.3}$$

And then

$$\begin{aligned} E(a,b) &= E^p(a,b)E^q(a,b) \\ &= \left[ae^{\frac{b}{L(a,b)}-1}\right]^p\left[be^{\frac{a}{L(a,b)}-1}\right]^q \\ &= a^pb^q\exp\left[\frac{qa+pb}{L(a,b)}-1\right]. \end{aligned}$$

It follows that (3.1) holds. This proof is completed. ∎

(3.1) contains many expressions of $E(a,b)$, for example:

*1)* Let $p=q=\frac{1}{2}$.we easily obtain:

$$E(a,b)=G(a,b)\exp\left[\frac{A(a,b)}{L(a,b)}-1\right], \tag{3.4}$$

which also can be simply denoted by

$$E=Ge^{\frac{A}{L}-1}. \tag{3.5}$$

*2)* Let $p=\dfrac{L-a}{b-a}, q=\dfrac{b-L}{b-a}$.We have:

$$E=a^{\frac{L-a}{b-a}}b^{\frac{b-L}{b-a}}$$

*3)* Let $p=-\frac{a^t}{b^t-a^t}, q=\frac{b^t}{b^t-a^t}$ with $t\neq 0$, by an easy operation, we have:

$$E(a,b)=E_t\exp\left[\frac{t-1}{t}\left(\frac{G^2}{L\cdot J_{t-1}}-1\right)\right], \tag{3.6}$$

where $E_t=E^{\frac{1}{t}}(a^t,b^t), G=\sqrt{ab}, J_{t-1}=J_{t-1}(a,b)$,

$$J_t(a,b)=\begin{cases} \frac{t(a^{t+1}-b^{t+1})}{(t+1)(a^t-b^t)}, & t\neq 0,-1; \\ L(a,b), & t=0; \\ \frac{G^2(a,b)}{L(a,b)}, & t=-1. \end{cases} \tag{3.7}$$

In (3.6), taking $t=2,\frac{1}{2},\frac{1}{3}$, we can get the following identities:

$$E(a,b) = \sqrt{E(a^2,b^2)}\exp\left[\frac{1}{2}\left(\frac{G^2}{L\cdot A}-1\right)\right], \tag{3.8}$$

$$E(a,b) = E^2(\sqrt{a},\sqrt{b})\exp\left(1-\frac{G}{L}\right), \tag{3.9}$$

$$E(a,b) = E^3(\sqrt[3]{a},\sqrt[3]{b})\exp^2\left(1-\frac{A^{\frac{1}{3}}_{\frac{1}{3}}G^{\frac{2}{3}}}{L}\right), \tag{3.10}$$

respectively.

**Remark 1.** *Replace $t$ with $\frac{1}{t}$, then we can obtain identity (1.9) from (3.6). In fact, identities (3.9) and (3.10) are just (1.7) and (1.8).*

*4)* Let $p = \frac{a^t}{b^t+a^t}, q = \frac{b^t}{b^t+a^t}$, by an easy operation, we have:

$$E(a,b) = Z_t \exp\left(\frac{G^2}{L\cdot \mathcal{L}_t} - 1\right), \tag{3.11}$$

where $Z_t = Z^{\frac{1}{t}}(a^t, b^t), \mathcal{L}_t = \mathcal{L}_t(a,b)$,

$$\mathcal{L}_t(a,b) = \frac{a^t + b^t}{a^{t-1} + b^{t-1}} \tag{3.12}$$

is called Lehmer mean.

In (3.11), taking $t = 2, 1, \frac{1}{2}, \frac{1}{3}$, we can get the following identities:

$$E(a,b) = \sqrt{Z(a^2,b^2)} \exp\left(\frac{AG^2}{L\cdot A_2^2} - 1\right), \tag{3.13}$$

$$E(a,b) = Z(a,b) \exp\left(\frac{G^2}{LA} - 1\right), \tag{3.14}$$

$$E(a,b) = Z^2(\sqrt{a},\sqrt{b}) \exp\left(\frac{G}{L} - 1\right), \tag{3.15}$$

$$E(a,b) = Z^3(\sqrt[3]{a},\sqrt[3]{b}) \exp^2\left(\frac{A_{\frac{2}{3}}^{\frac{2}{3}} G^{\frac{2}{3}}}{LA_{\frac{1}{3}}^{\frac{1}{3}}} - 1\right), \tag{3.16}$$

respectively.

*5)* taking $\frac{1}{3}$-th power of two sides of (3.4) and $\frac{2}{3}$-th power of two sides of (3.15), and then let them multiply each other, we get

$$E(a,b) = G^{\frac{1}{3}}(a,b) Z_{\frac{1}{2}}^{\frac{2}{3}}(a,b) \exp\left[\frac{\frac{A(a,b)+2G(a,b)}{3}}{L(a,b)} - 1\right], \tag{3.17}$$

or concisely denoted by

$$E = G^{\frac{1}{3}} Z_{\frac{1}{2}}^{\frac{2}{3}} \exp\left[\frac{A+2G}{3L} - 1\right]. \tag{3.18}$$

In addition, substituting the right side of (3.4) for the left side of (3.15), then we get

$$E(a,b) = G(a,b)\exp\left(\frac{A(a,b)}{L(a,b)} - 1\right) = Z^2(\sqrt{a},\sqrt{b})\exp\left(\frac{G(a,b)}{L(a,b)} - 1\right).$$

It follows that the following Corollary is valid.

**Corollary 1.** *For positive numbers $a$ and $b$, there is*

$$Z^2(\sqrt{a},\sqrt{b}) = G(a,b) e^{\frac{A(a,b)-G(a,b)}{L(a,b)}}, \tag{3.19}$$

*which can be concisely denoted by*

$$Z_{\frac{1}{2}} = G e^{\frac{A-G}{L}}. \tag{3.20}$$

Applying (3.4), for $p \neq q$,

$$\begin{aligned}\mathcal{H}_E(a,b;p,q) &= \left(\frac{E(a^p,b^p)}{E(a^q,b^q)}\right)^{\frac{1}{p-q}} = \left\{\frac{G(a^p,b^p)\exp\left[\frac{A(a^p,b^p)}{L(a^p,b^p)}-1\right]}{G(a^q,b^q)\exp\left[\frac{A(a^q,b^q)}{L(a^q,b^q)}-1\right]}\right\}^{\frac{1}{p-q}} \\ &= G(a,b)\exp\left\{\frac{1}{p-q}\left[\frac{A(a^p,b^p)}{L(a^p,b^p)}-\frac{A(a^q,b^q)}{L(a^q,b^q)}\right]\right\}.\end{aligned}$$

Let

$$\mathcal{E}(p,q;a,b) = \frac{1}{p-q}\left[\frac{A(a^p,b^p)}{L(a^p,b^p)}-\frac{A(a^q,b^q)}{L(a^q,b^q)}\right]. \tag{3.21}$$

Then the part 1), 2) and 3) of Theorem 1 can be restated as follows:

**Corollary 2.** *1) $\mathcal{E}(p,q;a,b)$ are strictly increasing in $p$ or $q$ on $(-\infty,+\infty)$.*
*2) $\mathcal{E}(p,q;a,b)$ are strictly concave with respect to either $p$ or $q$ on $(0,+\infty)$, and convex on $(-\infty,0)$.*
*3) $\mathcal{E}(p,1-p;a,b)$ are strictly increasing in $p$ on $(-\infty,\frac{1}{2})$, and decreasing on $(\frac{1}{2},+\infty)$.*

Observe that

$$\begin{aligned}\ln G_{E,t}(a,b) &= \ln Y^{\frac{1}{t}}(a^t,b^t) = \frac{1}{t}\ln E(a^t,b^t)\exp[1-\frac{G^2(a^t,b^t)}{L^2(a^t,b^t)}] \\ &= \frac{1}{t}\left[\ln G(a^t,b^t)+\frac{A(a^t,b^t)}{L(a^t,b^t)}-1+1-\frac{G^2(a^t,b^t)}{L^2(a^t,b^t)}\right] \\ &= \ln G(a,b)+\frac{1}{t}\left[\frac{A(a^t,b^t)}{L(a^t,b^t)}-\frac{G^2(a^t,b^t)}{L^2(a^t,b^t)}\right],\end{aligned}$$

substituting for (1.6), after rearranging, (1.6) is transformed as

$$\begin{aligned}&\frac{2}{p+q}\left[\frac{A(a^{\frac{p+q}{2}},b^{\frac{p+q}{2}})}{L(a^{\frac{p+q}{2}},b^{\frac{p+q}{2}})}-\frac{G^2(a^{\frac{p+q}{2}},b^{\frac{p+q}{2}})}{L^2(a^{\frac{p+q}{2}},b^{\frac{p+q}{2}})}\right] > \mathcal{E}(p,q;a,b) > \\ &\frac{1}{2p}\left[\frac{A(a^p,b^p)}{L(a^p,b^p)}-\frac{G^2(a^p,b^p)}{L^2(a^p,b^p)}\right]+\frac{1}{2q}\left[\frac{A(a^q,b^q)}{L(a^q,b^q)}-\frac{G^2(a^q,b^q)}{L^2(a^q,b^q)}\right].\end{aligned} \tag{3.22}$$

And then part 4) of Theorem 1 can be restated as follows:

**Corollary 3.** *If $p+q>0$ with $p\neq q$, then inequality (3.22) are valid. Inequalities (3.22) reversed if $p+q<0$ with $p\neq q$.*

## 4. Some Applications

**Example 1.** *By identity (3.1) using the well-known inequalities* $e^x > 1+x$ *for* $x \in \mathbb{R}$ *with* $x \neq 0$ *and* $e^x > 1 + x + \frac{x^2}{2}$ *for* $x > 0$, *we get immediately:*

$$E(a,b) > a^p b^q \frac{pb+qa}{L(a,b)}, \tag{4.1}$$

$$E(a,b) > \frac{1}{2} a^p b^q \left[\frac{(pb+qa)^2}{L^2(a,b)} + 1\right], \tag{4.2}$$

*which are rewritten as:*

$$L(a,b)E(a,b) > a^p b^q (pb+qa), \tag{4.3}$$

$$L^2(a,b)E(a,b) > \frac{1}{2} a^p b^q \left[(pb+qa)^2 + L^2(a,b)\right] \tag{4.4}$$

*where* $p, q \in \mathbb{R}$ *with* $p + q = 1$.

*1)* Let $p = q = \frac{1}{2}$. We have

$$LE > AG, \tag{4.5}$$

$$L^2 E > \frac{1}{2} G \left[A^2 + L^2\right]. \tag{4.6}$$

*2)* Let $p = -\frac{a^t}{b^t - a^t}, q = \frac{b^t}{b^t - a^t}$ with $t \neq 0, 1$. We have

$$L J_{t-1} E^{\frac{t}{t-1}} > E_t^{\frac{t}{t-1}} G^2, \tag{4.7}$$

$$\left(L J_{t-1}\right)^2 E^{\frac{t}{t-1}} > E_t^{\frac{t}{t-1}} \left[G^2 + \left(L J_{t-1}\right)^2\right]. \tag{4.8}$$

In particular, for $t = \frac{1}{2}$, there are

$$L E_{\frac{1}{2}} > EG, \tag{4.9}$$

$$L^2 E_{\frac{1}{2}} > \frac{1}{2} E \left[G^2 + L^2\right]. \tag{4.10}$$

*3)* Let $p = \frac{a^t}{b^t + a^t}, q = \frac{b^t}{b^t + a^t}$. We have

$$L \mathcal{L}_t E > Z_t G^2, \tag{4.11}$$

$$\left(L \mathcal{L}_t\right)^2 E > \frac{1}{2} Z_t \left[G^2 + \left(L \mathcal{L}_t\right)^2\right]. \tag{4.12}$$

In particular, for $t = \frac{1}{2}$, there are

$$LE > Z_{\frac{1}{2}} G, \tag{4.13}$$

$$L^2 E > \frac{1}{2} Z_{\frac{1}{2}} \left[G^2 + L^2\right]. \tag{4.14}$$

**Example 2.** *By identity (3.1,* $E(a,b)$ *is comparable to* $a^p b^q$ *if and only if* $L(a,b)$ *is comparable to* $pb + aq$.

*1) For* $p = -\frac{a^t}{b^t - a^t}, q = \frac{b^t}{b^t - a^t}$, *since* $E_t = E^{\frac{1}{t}}(a^t, b^t)$ *is increasing in* $t$ *on interval* $(-\infty, +\infty)$, *so* $E > (<) E_t$ *if* $t > (<) 1$, *it follows from (3.6) that*

$$\frac{t-1}{t}\left(\frac{G^2}{L \cdot J_{t-1}} - 1\right) \begin{cases} < 0, & \textit{if } t > 1; \\ > 0, & \textit{if } 0 < t < 1; \\ > 0, & \textit{if } t < 0. \end{cases}$$

*i.e.*

$$L > \frac{G^2}{J_{t-1}} \text{ if } t > 0. \tag{4.15}$$

*Inequality (4.15) is reversed if $t < 0$.*

2) *For (3.16), it follows from (2.6) that $A_{\frac{2}{3}}^{\frac{2}{3}} G^{\frac{2}{3}}/(LA_{\frac{1}{3}}^{\frac{1}{3}}) - 1 < 0$, i.e.*

$$L > \frac{a^{\frac{2}{3}} + b^{\frac{2}{3}}}{a^{\frac{1}{3}} + b^{\frac{1}{3}}}(ab)^{\frac{1}{3}} = \frac{ab^{\frac{1}{3}} + ba^{\frac{1}{3}}}{a^{\frac{1}{3}} + b^{\frac{1}{3}}}, \tag{4.16}$$

*which was presented first by J. Karamata ([4]). Here give another proof of it.*

3) *In the same way, for (3.20), it follows from well-known inequality*

$$L < \frac{A + 2G}{3} \tag{4.17}$$

*that*

$$E > G^{\frac{1}{3}} Z_{\frac{1}{2}}^{\frac{2}{3}}, \tag{4.18}$$

*which is stronger than inequality $E > Z^{\frac{1}{3}} G^{\frac{2}{3}}$.*

**Example 3** (A left approximation of Gauss AGM [1, 2, 3])**.** *For (3.18), from (2.5) it follows that $Z_{\frac{1}{2}} = Ge^{\frac{A-G}{L}} > A$, which can be transformed as*

$$L < \frac{A - G}{\ln A - \ln G}. \tag{4.19}$$

*Let*

$$a_{n+1} = \frac{a_n + b_n}{2}, b_{n+1} = \sqrt{a_n b_n}, n = 0, 1, 2, \cdots \tag{4.20}$$

*with $a_0 = a > 0, b_0 = b > 0$. Then by (4.19) there is*

$$L(a_n, b_n) < L(a_{n+1}, b_{n+1}). \tag{4.21}$$

*It is easy to prove that sequence $\{a_n\}$ is monotone decreasing and bounded, while $\{b_n\}$ is monotone increasing and bounded. Then their limits both exist, and equal by (4.20), and might as well set $\mu_{AG} = \mu_{AG}(a, b)$. Thus we have*

$$\begin{aligned} G &< \sqrt{AG} < \sqrt{\sqrt{AG}\frac{A+G}{2}} \cdots < b_n < \cdots < \mu_{AG} \\ &< \cdots < a_n \cdots < \left(\frac{\sqrt{A} + \sqrt{G}}{2}\right)^2 < \frac{A+G}{2} < A. \end{aligned} \tag{4.22}$$

*On the other hand, by (4.21) sequence $\{L(a_n, b_n)\}$ is monotone increasing, and bounded above because $L(a_n, b_n) < \frac{a_n + b_n}{2} = a_{n+1} < a$. It follows that the limit of sequence $\{L(a_n, b_n)\}$ exists.*

*According to continuity of $L(x, y)$ on $\mathbb{R}_+ \times \mathbb{R}_+$, we have*

$$\lim_{n\to\infty} L(a_n, b_n) = L(\lim_{n\to\infty} a_n, \lim_{n\to\infty} b_n) = L(\mu_{AG}, \mu_{AG}) = \mu_{AG}. \tag{4.23}$$

*And then it follows from (4.21) that*

$$L(a, b) < L(A, G) < \cdots < L(a_n, b_n) < \cdots < \mu_{AG}. \tag{4.24}$$

*It is another left approximation of Gauss AGM that more precise than* $\{a_n\}$*.*

**Remark 2.** *By (4.19) and well-known Lin Tong-po inequality, we can obtain a new inequality regarding* $L$*,* $A$ *and* $G$*:*

$$L < \left( \frac{\sqrt[3]{A} + \sqrt[3]{G}}{2} \right)^3 . \tag{4.25}$$

**Example 4** (A new compound mean EGM and inequalities)**.** *That (3.5) can be rewritten as*

$$\ln E - \ln G = \frac{A}{L} - 1 = \frac{A - L}{L}; \tag{4.26}$$

*On the other hand, inequality (1.4) can be rewritten as*

$$E - G < A - L. \tag{4.27}$$

*It follows from (4.26) and (4.27) that*

$$L(E, G) < L. \tag{4.28}$$

*Let*

$$c_{n+1} = E(c_n, d_n), d_{n+1} = \sqrt{c_n d_n} \tag{4.29}$$

*with* $c_0 = a > 0, d_0 = b > 0$*. Then by (4.28) there is*

$$L(c_{n+1}, d_{n+1}) < L(c_n, d_n). \tag{4.30}$$

*First, using the well-known inequality* $\sqrt{ab} < E(a, b) < \frac{a+b}{2}$*, we have*

$$\begin{aligned} c_{n+1} &= E(c_n, d_n) > \sqrt{c_n d_n} = d_{n+1}, \\ c_{n+1} - c_n &= E(c_n, d_n) - c_n < \frac{c_n + d_n}{2} - c_n = \frac{d_n - c_n}{2} < 0, \\ d_{n+1} - d_n &= \sqrt{c_n d_n} - d_n = \sqrt{d_n}(\sqrt{c_n} - \sqrt{d_n}) > 0, \end{aligned}$$

*which implies sequence* $\{c_n\}$ *is monotone decreasing and* $\{d_n\}$ *is monotone decreasing. Hence*

$$\sqrt{ab} = d_1 < d_n < c_n < c_1 = E(a, b), \tag{4.31}$$

*which show that sequence* $\{c_n\}$ *and* $\{d_n\}$ *are bounded. It follows that limit of sequence* $\{c_n\}$ *and* $\{d_n\}$ *both exist.*

*Second, by (4.29) we have* $\lim_{n\to\infty} c_n = \lim_{n\to\infty} d_n$*, and might as well set* $\mu_{EG} = \mu_{EG}(a, b)$*. Thus we have*

$$\begin{aligned} G &< \sqrt{EG} < \sqrt{\sqrt{EG} E(E, G)} \cdots < d_n < \cdots < \mu_{EG} \\ &< \cdots < c_n \cdots < E(\sqrt{EG}, E(E, G)) < E(E, G) < E. \end{aligned} \tag{4.32}$$

*Third, by (4.30) sequence* $\{L(c_n, d_n)\}$ *is monotone decreasing and bounded because*

$$\sqrt{ab} = d_1 < d_{n+1} = \sqrt{c_n d_n} < L(c_n, d_n) < E(c_n, d_n) = c_{n+1} < c_1 = E(a, b).$$

*It follows that the limit of sequence* $\{L(c_n, d_n)\}$ *exists.*

*According to continuity of* $L(x, y)$ *on* $\mathbb{R}_+ \times \mathbb{R}_+$*, we have*

$$\lim_{n\to\infty} L(c_n, d_n) = L(\lim_{n\to\infty} c_n, \lim_{n\to\infty} d_n) = L(\mu_{EG}, \mu_{EG}) = \mu_{EG}. \tag{4.33}$$

*And then it follows from (4.30) that*

$$L(a,b) > L(E,G) > \cdots > L(c_n,d_n) > \cdots > \mu_{EG}. \tag{4.34}$$

*It is another right approximation of EGM that more precise than* $\{c_n\}$.

**Remark 3.** *By (4.22), (4.24), (4.32) and (4.34), we obtain immediately the following inequality chain:*

$$\begin{aligned}\sqrt{EG} &< \mu_{EG} < \cdots < L(c_n,d_n) < \cdots < L(E,G) < L(a,b) \\ &< L(A,G) < \cdots < L(a_n,b_n) < \cdots < \mu_{AG} < \frac{A+G}{2}\end{aligned} \tag{4.35}$$

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