ON THE MONOTONICITY AND LOG-CONVEXITY OF A FOUR-PARAMETER HOMOGENEOUS MEAN

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Abstract. A four-parameter homogeneous mean $\boldsymbol{F}(p,q;r,s;a,b)$ is defined by another approach. The criterion for monotonicity and logarithmically convexity of which are presented, and two refined two-parameter inequality's chains concerning some classical mean values are deduced.

1. Introduction

The so-called two-parameter mean or extended mean values between two unequal positive numbers x and y were defined first by K.B. Stolarsky [10] as

$$E(r,s;x,y)=\begin{cases}\left(\dfrac{s(x^r-y^r)}{r(x^s-y^s)}\right)^{\frac{1}{r-s}}, & r\neq s, rs\neq 0;\\ \left(\dfrac{x^r-y^r}{r(\ln x-\ln y)}\right)^{\frac{1}{r}}, & r\neq 0, s=0;\\ \left(\dfrac{x^s-y^s}{s(\ln x-\ln y)}\right)^{\frac{1}{s}}, & r=0, s\neq 0;\\ \exp\left(\dfrac{x^r\ln x-y^r\ln y}{x^r-y^r}-\dfrac{1}{r}\right), & r=s\neq 0;\\ \sqrt{xy}, & r=s=0.\end{cases} \tag{1.1}$$

It contains many mean values, for instance:

$$E(1,0;x,y) = L(x,y)=\begin{cases}\frac{x-y}{\ln x-\ln y}, & x\neq y;\\ x, & x=y.\end{cases} \tag{1.2}$$

$$E(1,1;x,y) = E(x,y)=\begin{cases}e^{-1}\left(\dfrac{x^x}{y^y}\right)^{\frac{1}{x-y}}, & x\neq y;\\ x, & x=y.\end{cases} \tag{1.3}$$

$$E(2,1;x,y) = A(x,y)=\frac{x+y}{2}. \tag{1.4}$$

$$E(\frac{3}{2},\frac{1}{2};x,y) = h(x,y)=\frac{x+\sqrt{xy}+y}{3}. \tag{1.5}$$

The monotonicity of $E(r,s;x,y)$ has been researched by K.B. Stolarsky [10], E. B. Leach and M. C. Sholander [7] and others also in [3, 8, 9, 19] using different ideas and simpler methods.

Feng Qi studied the log-convexity for the parameters of the extended mean in [9], and pointed out the two-parameters mean is a log-concave function with respect to either parameter r or s on interval $(0,+\infty)$ and is a log-convex function on interval $(-\infty,0)$.

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In [13], Alfred Witkowski considered more general means defined by

$$R(u,v;r,s;x,y)=\left[\frac{E(u,v;x^r,y^r)}{E(u,v;x^s,y^s)}\right]^{\frac{1}{r-s}} \tag{1.6}$$

further and the following results for the monotonicity of R were obtained:

Theorem 1. *(Corollary 4 in [13]) R increases in r and s if $u+v>0$ and decreases otherwise.*

Theorem 2. *(Corollary 5 in [13]) R increases in u and v if $r+s>0$ and decreases otherwise.*

On the other hand, the extended mean was generalized to two-parameter homogeneous functions in [15, 16]. That is:

Definition 1. *Assume $f\colon \mathbb{U}(\subseteq \mathbb{R}_+\times\mathbb{R}_+)\to\mathbb{R}_+$ is an n-order homogeneous function for variables x and y, and is continuous and 1st partial derivatives exist, $(a,b)\in\mathbb{R}_+\times\mathbb{R}_+$ with $a\neq b$, $(p,q)\in\mathbb{R}\times\mathbb{R}$.*

If $(1,1)\notin\mathbb{U}$, then define that

$$\mathcal{H}_f(p,q;a,b) = \left[\frac{f(a^p,b^p)}{f(a^q,b^q)}\right]^{\frac{1}{p-q}} (p\neq q, pq\neq 0), \tag{1.7}$$

$$\mathcal{H}_f(p,p;a,b) = \lim_{q\to p}\mathcal{H}_f(a,b;p,q)=G_{f,p}(p=q\neq 0), \tag{1.8}$$

where

$$G_{f,p}=G_f^{\frac{1}{p}}(a^p,b^p),\quad G_f(x,y)=\exp\left[\frac{xf_x(x,y)\ln x+yf_y(x,y)\ln y}{f(x,y)}\right], \tag{1.9}$$

$f_x(x,y)$ and $f_y(x,y)$ denote partial derivatives with respect to 1st and 2nd variable of $f(x,y)$ respectively.

If $(1,1)\in\mathbb{U}$, then define further

$$\mathcal{H}_f(p,0;a,b) = \left[\frac{f(a^p,b^p)}{f(1,1)}\right]^{\frac{1}{p}} (p\neq 0, q=0), \tag{1.10}$$

$$\mathcal{H}_f(0,q;a,b) = \left[\frac{f(a^q,b^q)}{f(1,1)}\right]^{\frac{1}{q}} (p=0, q\neq 0), \tag{1.11}$$

$$\mathcal{H}_f(0,0;a,b) = \lim_{p\to 0}\mathcal{H}_f(a,b;p,0)=a^{\frac{f_x(1,1)}{f(1,1)}}b^{\frac{f_y(1,1)}{f(1,1)}}(p=q=0). \tag{1.12}$$

When $f(x,y)=L(x,y)$, we can get two-parameter logarithmic mean $\mathcal{H}_L(p,q;a,b)$, which is just equal to extended mean $E(p,q;a,b)$ defined by (1.1). For avoiding confusion, the extended mean will be called two-parameter logarithmic mean, and denote by $\mathcal{H}_L(p,q;a,b)$ or $\mathcal{H}_L(p,q)$ or $\mathcal{H}_L$ in what follows.

Concerning the monotonicity and log-convexity of the two-parameter homogeneous functions, there are the following results:

Theorem 3. [15, 16]*Let $f(x,y)$ be a positive n-order homogenous function defined on $\mathbb{U}(\subseteq\mathbb{R}_+\times\mathbb{R}_+)$ and be 2nd differentiable. If $I_1=(\ln f)_{xy}<(>)0$, then $\mathcal{H}_f(p,q)$ is strictly increasing (decreasing) in either p or q on $(-\infty,0)\cup(0,+\infty)$.*

Theorem 4. [17, 18]*Let $f(x,y)$ be a positive n-order homogenous function defined on $\mathbb{U}(\subseteq\mathbb{R}_+\times\mathbb{R}_+)$ and be 3rd differentiable. If*

$$J=(x-y)(xI_1)_x<(>)0,\ \text{where } I_1=(\ln f)_{xy}, \tag{1.13}$$

then $\mathcal{H}_f(p,q)$ is strictly log-convex (log-concave) in either p or q on $(0,+\infty)$, and log-concave (log-convex) on $(-\infty,0)$.

By the above theorems we have

Corollary 1. *The conditions are the same as in Theorem 3. If (1.13) holds, then* $\mathcal{H}_f(p,1-p)$ *is strictly decreasing (increasing) in* p *on* $(0,\frac{1}{2})$*, increasing (decreasing) on* $(\frac{1}{2},1)$*.*

If $f(x,y)$ *is symmetric with respect to* x *and* y *further, then the above monotone interval can be extended from* $(0,\frac{1}{2})$ *to* $(-\infty,0)\cup(0,\frac{1}{2})$ *and* $(\frac{1}{2},1)$ *to* $(\frac{1}{2},1)\cup(1,+\infty)$*, respectively.*

Corollary 2. *The conditions are the same as Theorem 3. If (1.13) holds, then for* $p,q\in(0,+\infty)$ *with* $p\neq q$*, there is*

$$G_{f,\frac{p+q}{2}} < (>)\mathcal{H}_f(p,q) < (>)\sqrt{G_{f,p}G_{f,q}}. \tag{1.14}$$

For $p,q\in(-\infty,0)$ *with* $p\neq q$*, inequality (1.14) is reversed.*

If $f(x,y)$ *is defined on* $\mathbb{R}_+\times\mathbb{R}_+$ *and is symmetric with respect to* x *and* y *further, then substituting* $p+q>0$ *for* $p,q\in(0,+\infty)$ *and* $p+q<0$ *for* $p,q\in(-\infty,0)$*, (1.14) is also true, respectively.*

As applications of the above results, we also have the following conclusions:

Conclusion 1. *For* $f(x,y)=L(x,y)$*,* $A(x,y)$*,* $E(x,y)$*, where* $x,y>0$ *with* $x\neq y$*, then*

1) $\mathcal{H}_f(p,q)$ *are strictly increasing in either* p *or* q *on* $(-\infty,+\infty)$*;*

2) $\mathcal{H}_f(p,q)$ *are strictly log-concave in either* p *or* q *on* $(0,+\infty)$*, and log-convex on* $(-\infty,0)$*;*

3) $\mathcal{H}_f(p,1-p)$ *are strictly increasing in* p *on* $(-\infty,\frac{1}{2})$*, and decreasing on* $(\frac{1}{2},+\infty)$*.*

4) If $p+q>0$*, then*

$$G_{f,\frac{p+q}{2}} > \mathcal{H}_f(p,q) > \sqrt{G_{f,p}G_{f,q}}. \tag{1.15}$$

Inequality (1.15) is reversed if $p+q<0$*.*

Conclusion 2. *For* $f(x,y)=D(x,y)=|x-y|$*, where* $x,y>0$ *with* $x\neq y$*, then*

1) $\mathcal{H}_D(p,q)$ *is strictly decreasing in either* p *or* q *on* $(-\infty,0)\cup(0,+\infty)$*;*

2) $\mathcal{H}_f(p,q)$ *is strictly log-concave in either* p *or* q *on* $(-\infty,0)$*, and log-convex on* $(0,+\infty)$*;*

3) $\mathcal{H}_D(p,1-p)$ *is strictly decreasing in* p *on* $(-\infty,0)\cup(0,\frac{1}{2})$*, and increasing on* $(\frac{1}{2},1)\cup(1,+\infty)$*;*

4) If $p,q\in(0,+\infty)$*, there is*

$$G_{D,\frac{p+q}{2}} < \mathcal{H}_D(p,q) < \sqrt{G_{D,p}G_{D,q}}. \tag{1.16}$$

Inequality (1.16) is reversed if $p,q\in(-\infty,0)$*.*

2. Main Results

Let us substitute $\mathcal{H}_L(r,s;x,y)$ for $f(x,y)$ in Definition 1, then $\mathcal{H}_f(p,q;a,b)$ is a mean of positive x and y with four parameters r,s,p and q, which is called four-parameter mean values. For expedience, we will adopt our notations to introduce the Definition.

Definition 2. *Assume* $(a,b)\in\mathbb{R}_+\times\mathbb{R}_+$ *with* $a\neq b$*,* $(p,q),(r,s)\in\mathbb{R}\times\mathbb{R}$*, then call* $\boldsymbol{F}(p,q;r,s;a,b)$ *four-parameter homogeneous mean, which is defined as follows:*

(2.1)

$$\boldsymbol{F}(p,q;r,s;a,b)=\left[\frac{L(a^{pr},b^{pr})}{L(a^{ps},b^{ps})}\frac{L(a^{qs},b^{qs})}{L(a^{qr},b^{qr})}\right]^{\frac{1}{(p-q)(r-s)}},\ \text{if } pqrs(p-q)(r-s)\neq 0,$$

or

$$(2.2)\quad \boldsymbol{F}(p,q;r,s;a,b)=\left[\frac{a^{pr}-b^{pr}}{a^{ps}-b^{ps}}\frac{a^{qs}-b^{qs}}{a^{qr}-b^{qr}}\right]^{\frac{1}{(p-q)(r-s)}},\text{if } pqrs(p-q)(r-s)\neq 0.$$

if $pqrs(p-q)(r-s)=0$, *then the* $\boldsymbol{F}(a,b;p,q;r,s)$ *are defined as its corresponding limits, for example:*

$$\begin{aligned}
\boldsymbol{F}(p,p;r,s;a,b) &= \lim_{q\to p}\boldsymbol{F}(a,b;p,q;r,s)=\left[\frac{E(a^{pr},b^{pr})}{E(a^{ps},b^{ps})}\right]^{\frac{1}{p(r-s)}},\ \text{if } prs(r-s)\neq 0, p=q,\\
\boldsymbol{F}(p,0;r,s;a,b) &= \lim_{q\to 0}\boldsymbol{F}(a,b;p,q;r,s)=\left[\frac{L(a^{pr},b^{pr})}{L(a^{ps},b^{ps})}\right]^{\frac{1}{p(r-s)}},\ \text{if } prs(r-s)\neq 0, q=0,\\
\boldsymbol{F}(0,0;r,s;a,b) &= \lim_{p\to 0}\boldsymbol{F}(a,b;p,0;r,s)=G(a,b),\ \text{if } rs(r-s)\neq 0, p=q=0,
\end{aligned}$$

where $L(x,y), E(x,y)$*are defined by (1.2), (1.3) respectively,* $G(a,b)=\sqrt{ab}$

In the case of not being confused, we set

$$\boldsymbol{F}(p,q;r,s;a,b)=\boldsymbol{F}(p,q)=\boldsymbol{F}(r,s)=\boldsymbol{F}(p,q;r,s)=\boldsymbol{F}(a,b)$$

The following properties of four-parameter mean values $\boldsymbol{F}(a,b;p,q;r,s)$ are verified easily:

Property 1 $\boldsymbol{F}(p,q;r,s;a,b)$ *are symmetric with respect to* a *and* b*, i.e.*

$$(2.3)\qquad \boldsymbol{F}(a,b)=\boldsymbol{F}(b,a);$$

Property 2 $\boldsymbol{F}(p,q;r,s;a,b)$ *are symmetric with respect to* p *and* q *, i.e.*

$$(2.4)\qquad \boldsymbol{F}(p,q)=\boldsymbol{F}(q,p);$$

Property 3 $\boldsymbol{F}(p,q;r,s;a,b)$ *are symmetric with respect to* r *and* s*, i.e.*

$$(2.5)\qquad \boldsymbol{F}(r,s)=\boldsymbol{F}(s,r);$$

Property 4 $\boldsymbol{F}(p,q;r,s;a,b)$ *are symmetric with respect to* (p,q) *and* (r,s)*, i.e.*

$$(2.6)\qquad \boldsymbol{F}(p,q;r,s)=\boldsymbol{F}(r,s;p,q).$$

Obviously, so long as the signs of I_1 and J are certain, then the monotonicity and log-convexity of $\mathcal{H}_f(p,q)$ with respect to either p or q are also certain with it. For example, for $f(x,y)=L(x,y)$, $A(x,y)$, $E(x,y)$, there are $I_1<0$, $J>0$, and then corresponding monotonicity and log-convexity of two-parameter homogeneous functions $\mathcal{H}_f(p,q)$ are confirmed.

Owing to that $\mathcal{H}_L(r,s;x,y)$ contain $L(x,y)$, $A(x,y)$ and $E(x,y)$, naturally, we could make conjecture on there are $I_1=(\ln f)_{xy}<0$, $J=(x-y)(xI_1)_x>0$ for $f(x,y)=\mathcal{H}_L(r,s;x,y)$. The purpose of this paper is to verify the conjecture, and get accordingly the following results on the monotonicity and log-convexity of $\mathcal{H}_f(p,q)$, where $f(x,y)=\mathcal{H}_L(r,s;x,y)$.

Theorem 5. *If* $r+s>(<)0$*, then* $\boldsymbol{F}(p,q;r,s;a,b)$ *are strictly increasing (decreasing) in either* p *or* q *on* $(-\infty,+\infty)$;

Theorem 6. *If* $r+s>(<)0$*, then* $\boldsymbol{F}(p,q;r,s;a,b)$ *are strictly log-concave (log-convex) in either* p *or* q *on* $(0,+\infty)$*, and log-convex (log-concave) on* $(-\infty,0)$;

Corollary 3. *If* $r+s>(<)0$*, then* $\boldsymbol{F}(p,1-p;r,s;a,b)$ *are strictly increasing (decreasing) in* p *on* $(-\infty,\frac{1}{2})$*, and decreasing (increasing) on* $(\frac{1}{2},+\infty)$.

Notice for $f(x,y) = \mathcal{H}_L(r,s;x,y)$, because

$$\begin{aligned} G_f(x,y) &= \exp\left[\frac{xf_x(x,y)\ln x + yf_y(x,y)\ln y}{f(x,y)}\right] \\ &= \exp\left[\frac{1}{r-s}\left(\frac{rx^r}{x^r-y^r} - \frac{sx^s}{x^s-y^s}\right)\ln x + \frac{1}{r-s}\left(-\frac{ry^r}{x^r-y^r} + \frac{sy^s}{x^s-y^s}\right)\ln y\right] \\ &= \exp^{\frac{1}{r-s}}\left[\left(\frac{x^r}{x^r-y^r}\ln x^r - \frac{y^r}{x^r-y^r}\ln y^r\right) - \left(\frac{x^s}{x^s-y^s}\ln x^s - \frac{y^s}{x^s-y^s}\ln y^s\right)\right] \\ &= \left[\frac{E(x^r,y^r)}{E(x^s,y^s)}\right]^{\frac{1}{r-s}}, \end{aligned}$$

by Theorem 6 and 2, we get

Corollary 4. *Suppose* $(p+q)(r+s) < 0$, *then*

$$G_{\mathcal{H}_L,\frac{p+q}{2}} < \boldsymbol{F}(p,q;r,s;a,b) < \sqrt{G_{\mathcal{H}_L,p}G_{\mathcal{H}_L,q}} \tag{2.7}$$

where $G_{\mathcal{H}_L,t} = G_{\mathcal{H}_L}^{\frac{1}{t}}(a^t,b^t)$, $G_{\mathcal{H}_L}(x,y) = \left[\frac{E(x^r,y^r)}{E(x^s,y^s)}\right]^{\frac{1}{r-s}}$, $E(x,y)$ *is defined by (1.3).*

Inequality (2.7) is reversed if $(p+q)(r+s) > 0$.

3. Lemmas

The following three lemmas are useful in proofs of the main results.

Lemma 1. *Suppose* $x, y > 0$ *with* $x \neq y$,*let*

$$K(t) = \begin{cases} x^t y^t \left[\dfrac{x^t - y^t}{t(x-y)}\right]^{-2}, & t \neq 0; \\ L^2(x,y), & t = 0. \end{cases} \tag{3.1}$$

then we have

1) $K(-t) = K(t)$;

2) $K(t)$ *is strictly increasing in* $(-\infty, 0)$, *and decreasing in* $(0, +\infty)$.

Proof. 1) An easy computation results in part 1) of the Lemma, of which details are omitted.

2) By directly calculations, we get

$$\begin{aligned} \frac{K'(t)}{K(t)} &= \ln x + \ln y - \frac{2(x^t\ln x - y^t \ln y)}{x^t - y^t} + \frac{2}{t} \\ &= \frac{2}{t}\left[\ln\sqrt{x^t y^t} - \left(\frac{x^t \ln x - y^t \ln y}{x^t - y^t} - 1\right)\right] \\ &= \frac{2}{t}\left[\ln G(x^t,y^t) - \ln E(x^t,y^t)\right]. \end{aligned}$$

By the well-known inequality $E(a,b) > \sqrt{ab}$, we can get part two of the Lemma immediately. ∎

The following Lemma is a well-known inequality [5], which will be used in proof of Lemma 3.

Lemma 2. *For positive numbers* a *and* b, *the following inequality holds:*

$$L(a,b) < \frac{A+2G}{3} = \frac{a + 4\sqrt{ab} + b}{6} \tag{3.2}$$

Lemma 3. *Suppose $x, y > 0$ with $x \neq y$,let*

$$N(t) = \begin{cases} x^t y^t \dfrac{x^t + y^t}{2} \left[\dfrac{x^t - y^t}{t(x-y)}\right]^{-3}, & t \neq 0; \\ L^3(x,y), & t = 0. \end{cases} \tag{3.3}$$

then we have

1) $N(-t) = N(t)$;

2) $N(t)$ is strictly increasing in $(-\infty, 0)$, and decreasing in $(0, +\infty)$.

Proof. 1) An easy computation results in part one, of which details are omitted.

2) By direct calculations, we get

$$\begin{aligned} \frac{N'(t)}{N(t)} &= \ln x + \ln y + \frac{x^t \ln x + y^t \ln y}{x^t + y^t} - \frac{3(x^t \ln x - y^t \ln y)}{x^t - y^t} + \frac{3}{t} \\ &= \left(1 + \frac{x^t}{x^t + y^t} - \frac{3x^t}{x^t - y^t}\right) \ln x + \left(1 + \frac{y^t}{x^t + y^t} + \frac{3y^t}{x^t - y^t}\right) \ln y + \frac{3}{t} \\ &= -\frac{x^{2t} + 4x^t y^t + y^{2t}}{x^{2t} - y^{2t}} \ln x + \frac{x^{2t} + 4x^t y^t + y^{2t}}{x^{2t} - y^{2t}} \ln y + \frac{3}{t} \\ &= \frac{3}{t} - \frac{x^{2t} + 4x^t y^t + y^{2t}}{x^{2t} - y^{2t}} (\ln x - \ln y) \\ &= \frac{3}{t} \frac{2t(\ln x - \ln y)}{x^{2t} - y^{2t}} \left[\frac{x^{2t} - y^{2t}}{2t(\ln x - \ln y)} - \frac{x^{2t} + 4x^t y^t + y^{2t}}{6}\right]. \end{aligned}$$

Substituting a, b for x^{2t}, y^{2t} in the above last one expression, then

$$\frac{N'(t)}{N(t)} = \frac{3}{t} L^{-1}(a,b) \left[L(a,b) - \frac{a + 4\sqrt{ab} + b}{6}\right], \tag{3.4}$$

in which $L(a,b) - \dfrac{a + 4\sqrt{ab} + b}{6} < 0$ by Lemma 2, and $L^{-1}(a,b) > 0$. Consequently, $N'(t) > 0$ if $t < 0$, and $N'(t) < 0$ if $t > 0$. The proof is completed. ∎

4. Proofs of Main Results

Since $\boldsymbol{F}(a,b;p,q;r,s) = \mathcal{H}_{\mathcal{H}_L}(a,b;p,q)$, where $\mathcal{H}_L = \mathcal{H}_L(r,s;x,y) = E(r,s;x,y)$ is defined by (1.3), it is enough to make certain the signs of $I_1 = (\ln \mathcal{H}_L)_{xy}$ and $J = (x-y)(xI_1)_x$.

Proof of Theorem 5. Let us observe that

$$\ln \mathcal{H}_L = \frac{1}{r-s} \left[\ln|s| + \ln|x^r - y^r| - \ln|r| - \ln|x^s - y^s|\right].$$

Through straightforward computations, we have

$$\begin{aligned} I_1 &= (\ln \mathcal{H}_L)_{xy} = \frac{1}{xy\,(r-s)} \left[\frac{r^2 x^r y^r}{(x^r - y^r)^2} - \frac{s^2 x^s y^s}{(x^s - y^s)^2}\right] \\ &= \frac{1}{xy\,(r-s)} \left[\frac{r^2 x^r y^r}{(x^r - y^r)^2} - \frac{s^2 x^s y^s}{(x^s - y^s)^2}\right] \\ &= \frac{1}{xy(x-y)^2} \frac{K(r) - K(s)}{r-s}. \end{aligned}$$

By Lemma 1, if $r > s > 0$, we have $\dfrac{K(r)-K(s)}{r-s} < 0$; If $r > -s > 0$, we have also $\dfrac{K(r)-K(s)}{r-s} = \dfrac{K(r)-K(-s)}{r+(-s)} < 0$. Thus $I_1 < 0$ if $r+s > 0$.Likewise $I_1 > 0$ if $r+s < 0$.

By Theorem 3, this proof is completed. ∎

proof of Theorem 6. Let us consider that

$$\begin{aligned} J &= (x-y)\,(xI_1)_x = \frac{x-y}{xy(r-s)}\left[-\frac{r^3x^ry^r\,(x^r+y^r)}{(x^r-y^r)^3}+\frac{s^3x^sy^s(x^s+y^s)}{(x^s-y^s)^3}\right] \\ &= \frac{-2}{xy(x-y)^2}\frac{N(r)-N(s)}{r-s}. \end{aligned}$$

By Lemma 3, if $r > s > 0$, we have $\dfrac{N(r)-N(s)}{r-s} < 0$; If $r > -s > 0$, we also have $\dfrac{N(r)-N(s)}{r-s} = \dfrac{N(r)-N(-s)}{r+(-s)} < 0$. Thus $J > 0$ if $r+s > 0$.Likewise $J < 0$ if $r+s < 0$.

Using Theorem 4, this completes the proof. ∎

5. Inequality's Chains for Two-parameter Means

The four-parameter homogeneous mean values $\boldsymbol{F}(p,q;r,s;a,b)$ contain a good many two-parameter means, for example: (see Table 1)

(p,q)	$\boldsymbol{F}(p,q;r,s;a,b)$	(p,q)	$\boldsymbol{F}(p,q;r,s;a,b)$
$(2,1)$	$\left(\dfrac{a^r+b^r}{a^s+b^s}\right)^{\frac{1}{r-s}}$	$(\frac{1}{2},\frac{1}{2})$	$\left[\dfrac{E(a^{\frac{r}{2}},b^{\frac{r}{2}})}{E(a^{\frac{s}{2}},b^{\frac{s}{2}})}\right]^{\frac{2}{r-s}}$
$(1,1)$	$\left[\dfrac{E(a^r,b^r)}{E(a^s,b^s)}\right]^{\frac{1}{r-s}}$	$(\frac{3}{4},\frac{1}{4})$	$\left(\dfrac{a^{\frac{r}{2}}+(\sqrt{ab})^{\frac{r}{2}}+b^{\frac{r}{2}}}{a^{\frac{s}{2}}+(\sqrt{ab})^{\frac{s}{2}}+b^{\frac{s}{2}}}\right)^{\frac{2}{r-s}}$
$(1,\frac{1}{2})$	$\left(\dfrac{a^{\frac{r}{2}}+b^{\frac{r}{2}}}{a^{\frac{s}{2}}+b^{\frac{s}{2}}}\right)^{\frac{2}{r-s}}$	$(\frac{2}{3},\frac{1}{3})$	$\left(\dfrac{a^{\frac{r}{3}}+b^{\frac{r}{3}}}{a^{\frac{s}{3}}+b^{\frac{s}{3}}}\right)^{\frac{3}{r-s}}$
$(0,1)$	$\left(\frac{s}{r}\dfrac{a^r-b^r}{a^s-b^s}\right)^{\frac{1}{r-s}}$	$(\frac{3}{2},-\frac{1}{2})$	$\left(\dfrac{a^r+(\sqrt{ab})^r+b^r}{a^s+(\sqrt{ab})^s+b^s}\right)^{\frac{1}{2(r-s)}}(\sqrt{ab})^{\frac{1}{2}}$
$(1,-1)$	$\sqrt{ab}$	$(2,-1)$	$\left(\dfrac{a^r+b^r}{a^s+b^s}\right)^{\frac{1}{3(r-s)}}(\sqrt{ab})^{\frac{2}{3}}$

%

Table 1. some familiar two-parameter mean values

Example 1. *By Theorem 5, we can get a series of inequalities in form of two-parameter. If* $r+s > 0$*, then*

$$\begin{aligned} &\boldsymbol{F}(1,-1;r,s;a,b) < \boldsymbol{F}(0,1;r,s;a,b) < \boldsymbol{F}(1,\tfrac{1}{2};r,s;a,b) \\ &\qquad < \boldsymbol{F}(1,1;r,s;a,b) < \boldsymbol{F}(2,1;r,s;a,b), \end{aligned} \tag{5.1}$$

i.e.

(5.2)
$$G<\left(\frac{s}{r}\frac{a^r-b^r}{a^s-b^s}\right)^{\frac{1}{r-s}}<\left(\frac{a^{\frac{r}{2}}+b^{\frac{r}{2}}}{a^{\frac{s}{2}}+b^{\frac{s}{2}}}\right)^{\frac{2}{r-s}}<\left[\frac{E(a^r,b^r)}{E(a^s,b^s)}\right]^{\frac{1}{r-s}}<\left(\frac{a^r+b^r}{a^s+b^s}\right)^{\frac{1}{r-s}},$$

which can be concisely denoted by:

(5.3)
$$G<\left[\frac{L(a^r,b^r)}{L(a^s,b^s)}\right]^{\frac{1}{r-s}}<\left[\frac{A(a^{\frac{r}{2}},b^{\frac{r}{2}})}{A(a^{\frac{s}{2}},b^{\frac{s}{2}})}\right]^{\frac{2}{r-s}}<\left[\frac{E(a^r,b^r)}{E(a^s,b^s)}\right]^{\frac{1}{r-s}}<\left[\frac{A(a^r,b^r)}{A(a^s,b^s)}\right]^{\frac{1}{r-s}},$$

where L, E, A are defined by (1.2)-(1.4).

Remark 1. *Inequality (5.2) or (5.3) is a generalization of the following inequalities*
$$G<L<\frac{A+G}{2}<E<A.$$

Example 2. *By Theorem 3, we can get another more refined inequalities. If* $r+s>0$*, then*

$$\begin{aligned}&\boldsymbol{F}(\tfrac{1}{2},\tfrac{1}{2};r,s;a,b)>\boldsymbol{F}(\tfrac{2}{3},\tfrac{1}{3};r,s;a,b)>\boldsymbol{F}(\tfrac{3}{4},\tfrac{1}{4};r,s;a,b)>\\&\boldsymbol{F}(1,0;r,s;a,b)>\boldsymbol{F}(\tfrac{3}{2},-\tfrac{1}{2};r,s;a,b)>\boldsymbol{F}(2,-1;r,s;a,b),\end{aligned}\tag{5.4}$$

i.e.

$$\begin{aligned}&\left[\tfrac{E(a^{\frac{r}{2}},b^{\frac{r}{2}})}{E(a^{\frac{s}{2}},b^{\frac{s}{2}})}\right]^{\frac{2}{r-s}}>\left(\frac{a^{\frac{r}{3}}+b^{\frac{r}{3}}}{a^{\frac{s}{3}}+b^{\frac{s}{3}}}\right)^{\frac{3}{r-s}}>\left(\frac{a^{\frac{r}{2}}+\sqrt{a^{\frac{r}{2}}b^{\frac{r}{2}}}+b^{\frac{r}{2}}}{a^{\frac{s}{2}}+\sqrt{a^{\frac{s}{2}}b^{\frac{s}{2}}}+b^{\frac{s}{2}}}\right)^{\frac{2}{r-s}}>\\&\left(\frac{s}{r}\frac{a^r-b^r}{a^s-b^s}\right)^{\frac{1}{r-s}}>\left(\frac{a^r+\sqrt{a^rb^r}+b^r}{a^s+\sqrt{a^sb^s}+b^s}\right)^{\frac{1}{2(r-s)}}\sqrt{G}>\left(\frac{a^r+b^r}{a^s+b^s}\right)^{\frac{1}{3(r-s)}}G^{\frac{2}{3}},\end{aligned}\tag{5.5}$$

which can be concisely denoted by

$$\begin{aligned}&\left[\frac{E(a^{\frac{r}{2}},b^{\frac{r}{2}})}{E(a^{\frac{s}{2}},b^{\frac{s}{2}})}\right]^{\frac{2}{r-s}}>\left[\frac{A(a^{\frac{r}{3}},b^{\frac{r}{3}})}{A(a^{\frac{s}{3}},b^{\frac{s}{3}})}\right]^{\frac{3}{r-s}}>\left[\frac{h(a^{\frac{r}{2}},b^{\frac{r}{2}})}{h(a^{\frac{s}{2}},b^{\frac{s}{2}})}\right]^{\frac{2}{r-s}}>\\&\left[\frac{L(a^r,b^r)}{L(a^s,b^s)}\right]^{\frac{1}{r-s}}>\left[\frac{h(a^r,b^r)}{h(a^s,b^s)}\right]^{\frac{1}{2(r-s)}}\sqrt{G}>\left[\frac{A(a^r,b^r)}{A(a^s,b^s)}\right]^{\frac{1}{3(r-s)}}G^{\frac{2}{3}},\end{aligned}\tag{5.6}$$

where $L(x,y),E(x,y)$ $A(x,y)$ *and* $h(x,y)$ *and are defined by (1.2)-(1.5), respectively.*

Example 3. *If replace a, b with* a^2,b^2*, then inequalities (5.6) can be rewritten as:*

$$\begin{aligned}&\left[\frac{E(a^r,b^r)}{E(a^s,b^s)}\right]^{\frac{1}{r-s}}>\left[\frac{A(a^{\frac{2r}{3}},b^{\frac{2r}{3}})}{A(a^{\frac{2s}{3}},b^{\frac{2s}{3}})}\right]^{\frac{3}{2(r-s)}}>\left[\frac{h(a^r,b^r)}{h(a^s,b^s)}\right]^{\frac{1}{r-s}}>\\&\left[\frac{L(a^{2r},b^{2r})}{L(a^{2s},b^{2s})}\right]^{\frac{1}{2(r-s)}}>\left[\frac{h(a^{2r},b^{2r})}{h(a^{2s},b^{2s})}\right]^{\frac{1}{4(r-s)}}\sqrt{G}>\left[\frac{A(a^{2r},b^{2r})}{A(a^{2s},b^{2s})}\right]^{\frac{1}{6(r-s)}}G^{\frac{2}{3}}.\end{aligned}\tag{5.7}$$

Remark 2. *Inequality (5.6) or (5.7) not only strengthen and generalize Lin Tong-po and Stolarsky inequality, but also unifies them into the same inequality's chain.*

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