SOME UPPER BOUNDS FOR THE PRODUCT $p_1p_2\cdots p_n$

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Abstract. In this note, using refined Mandl's inequality, Robin's inequality and a refinements of the AGM inequality, we find some upper bounds for the product $p_1p_2\cdots p_n$.

As usual, let p_n be the n^{th} prime. The Mandl's inequality (see [1] and [5]) asserts that for every $n\geq 9$, we have:

$$\frac{1}{n}\sum_{i=1}^{n}p_i<\frac{p_n}{2}. \tag{1.1}$$

Robin's inequality (see [1], page 51) gives a lower bound for the average $\frac{1}{n}\sum_{i=1}^{n}p_i$; for every $n\geq 2$, it asserts that:

$$p_{\lfloor\frac{n}{2}\rfloor}\leq\frac{1}{n}\sum_{i=1}^{n}p_i. \tag{1.2}$$

A refinement of Mandl's inequality has been obtained in [2], as follows:

$$\frac{1}{n}\sum_{i=1}^{n}p_i<\frac{p_n}{2}-\frac{n}{14}\qquad(n\geq 10). \tag{1.3}$$

Using (1.3) and the AGM Inequality [3], we obtain:

$$p_1p_2\cdots p_n<\left(\frac{p_n}{2}-\frac{n}{14}\right)^n\qquad(n\geq 10). \tag{1.4}$$

Note that (1.4) holds also for $5\leq n\leq 9$. This yields an upper bound for the product $p_1p_2\cdots p_n$, which has been appeared in [2], already. In this short note, we use a refinement of the AGM inequality to get some better bounds. In [4], Rooin shows that for any non-negative real numbers $x_1\leq x_2\leq\cdots\leq x_n$, we have:

$$A_n-G_n\geq\frac{1}{n}\sum_{k=2}^{n}A_{n-1}^{\frac{n-k}{n}}(x_n^{\frac{1}{n}}-A_{n-1}^{\frac{1}{n}})^k\geq 0, \tag{1.5}$$

in which:

$$A_n=\frac{1}{n}\sum_{i=1}^{n}x_i,$$

and

$$G_n=\sqrt[n]{\prod_{i=1}^{n}x_i}.$$

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Applying (1.5) on $p_1 < p_2 < \cdots < p_n$, and using relations (1.2) and (1.3), for every $n \geq 10$, we obtain:

$$p_1 p_2 \cdots p_n < \left\{ \left(\frac{p_n}{2} - \frac{n}{14} \right) - \Omega(n) \right\}^n, \tag{1.6}$$

in which,

$$\Omega(n) = \frac{1}{n} \sum_{k=2}^{n} p_{\lfloor \frac{n-1}{2} \rfloor}^{\frac{n-k}{n}} \left\{ p_n^{\frac{1}{n}} - \left(\frac{p_n}{2} - \frac{n}{14} \right)^{\frac{1}{n}} \right\}^k > 0.$$

In fact, all members under summation are positive. So,

$$\Omega(n) > \frac{1}{n} \left\{ p_n^{\frac{1}{n}} - \left(\frac{p_n}{2} - \frac{n}{14} \right)^{\frac{1}{n}} \right\}^n > \frac{p_n}{2n} \left(2^{\frac{1}{n}} - 1 \right)^n.$$

Using this bound for $\Omega(n)$ and considering (1.6), for every $n \geq 10$, we obtain:

$$p_1 p_2 \cdots p_n < \left\{ \frac{p_n}{2} \left(1 - \frac{\left(2^{\frac{1}{n}} - 1 \right)^n}{n} \right) - \frac{n}{14} \right\}^n.$$

References

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