ON REFINEMENTS AND EXTENSIONS OF LOG-CONVEXITY FOR TWO-PARAMETER HOMOGENEOUS FUNCTIONS

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Abstract. In this paper, the log-convexity of two-parameter homogeneous functions and its corollaries in [4] are refined and extended. As applications, some conclusions about L, A, E and D are also strengthened and generalized.

1. Introduction and Main Results

The conception of two-parameter homogeneous functions was established by [3]:

Definition 1. *Assume $f\colon \mathbb{U}(\subseteqq \mathbb{R}_+ \times \mathbb{R}_+) \to \mathbb{R}_+$ is a homogeneous function for variable x and y, and is continuous and exist 1st partial derivative, $(a,b) \in \mathbb{R}_+ \times \mathbb{R}_+$ with $a \neq b$, $(p,q) \in \mathbb{R} \times \mathbb{R}$. If $(1,1) \notin \mathbb{U}$, then define that*

$$\mathcal{H}_f(a,b;p,q) = \left[\frac{f(a^p,b^p)}{f(a^q,b^q)}\right]^{\frac{1}{p-q}} (p \neq q, pq \neq 0), \tag{1.1}$$

$$\mathcal{H}_f(a,b;p,p) = \lim_{q\to p}\mathcal{H}_f(a,b;p,q) = G_f^{\frac{1}{p}}(a^p,b^p)(p = q \neq 0), \tag{1.2}$$

where

$$G_f(x,y) = \exp\left[\frac{xf_x(x,y)\ln x + yf_y(x,y)\ln y}{f(x,y)}\right], \tag{1.3}$$

$f_x(x,y)$ and $f_y(x,y)$ denote a partial derivative with respect to 1st and 2nd variable of $f(x,y)$ respectively.

If $(1,1) \in \mathbb{U}$, then define further

$$\mathcal{H}_f(a,b;p,0) = \left[\frac{f(a^p,b^p)}{f(1,1)}\right]^{\frac{1}{p}} (p \neq 0, q = 0), \tag{1.4}$$

$$\mathcal{H}_f(a,b;0,q) = \left[\frac{f(a^q,b^q)}{f(1,1)}\right]^{\frac{1}{q}} (p = 0, q \neq 0), \tag{1.5}$$

$$\mathcal{H}_f(a,b;0,0) = \lim_{p\to 0}\mathcal{H}_f(a,b;p,0) = G_{f,0}(a,b)(p = q = 0). \tag{1.6}$$

Date: May 26, 2005.

2000 *Mathematics Subject Classification.* Primary 26B25, 26D07; Secondary 26E60, 26A48 .

Key words and phrases. log-convexity, two-homogeneous function, mean, inequality.

In the case of not being confused, we set

$$\mathcal{H}_f = \mathcal{H}_f(p,q) = \mathcal{H}_f(a,b;p,q), \tag{1.7}$$

$$G_{f,p} = G_{f,p}(a,b) = G_f^{\frac{1}{p}}(a^p,b^p) = \mathcal{H}_f(p,p). \tag{1.8}$$

The author studied the log-convexity of $\mathcal{H}_f(a,b;p,q)$ in [4], and got the following results:

Theorem 1. *Let $f(x,y)$ be a positive n-order homogenous function defined on $\mathbb{U}(\subseteqq\mathbb{R}_+\times\mathbb{R}_+)$, and be 3-time differentiable. If*

$$J = (x-y)(xI_1)_x < (>)0, \text{ where } I_1 = (\ln f)_{xy}, \tag{1.9}$$

then when $p,q\in(0,+\infty)$, $\mathcal{H}_f(p,q)$ is logarithmically convex (concave) strictly for p or q respectively; while $p,q\in(-\infty,0)$, $\mathcal{H}_f(p,q)$ is logarithmically concave (convex) strictly for p or q respectively.

Corollary 1. *The conditions are the same as in Theorem 1. If (1.9) holds then $\mathcal{H}_f(p,1-p)$ is strictly monotone decreasing (increasing) in $p\in(0,\frac{1}{2})$, strictly monotone increasing (decreasing) in $p\in(\frac{1}{2},1)$.*

Corollary 2. *The conditions are the same as in Theorem 1. If (1.9) holds, then for $p,q\in(0,+\infty)$ with $p\neq q$, there is*

$$G_{f,\frac{p+q}{2}} < (>)\mathcal{H}_f(p,q) < (>)\sqrt{G_{f,p}G_{f,q}}, \tag{1.10}$$

where $G_{f,p}$ is defined by (1.8)

For $p,q\in(-\infty,0)$ with $p\neq q$, the inequality (1.10) reverses.

The aim of this paper is to refine and generalize the above results, which are stated as follows:

Theorem 2 (A Refinement of Theorem 1)**.** *The conditions are the same as in Theorem 1. If (1.9) holds, then $\mathcal{H}_f(p,q)$ is strictly log-convex (log-concave) with respect to either p or q on $(0,+\infty)$, and log-concave (log-convex) on $(-\infty,0)$.*

Remark 1. *This is an extension of Feng Qi's result on the log-convexity of extended mean values (see* [2]*).*

Applying Theorem 1, Corollary 1 can be refined as:

Corollary 3 (An Extension of Corollary 1)**.** *The conditions are the same as Theorem 1's, and $f(x,y)$ is symmetric with respect to x and y further. If (1.9) holds, then $\mathcal{H}_f(p,1-p)$ is strictly decreasing (increasing) in p on $(-\infty,0)\cup(0,\frac{1}{2})$, increasing (decreasing) on $(\frac{1}{2},1)\cup(1,+\infty)$.*

Corollary 4 (An Extension of Corollary 2)**.** *The conditions are the same as in Theorem 1, and $\mathbb{U}$=$\mathbb{R}_+\times\mathbb{R}_+$ and $f(x,y)$ is symmetric with respect to x and y further. If (1.9) holds, then (1.10) is also true for $p+q>0$ with $p\neq q$, (1.10) reverses for $p+q<0$ with $p\neq q$.*

2. Properties and Lemmas

The following properties of $\mathcal{H}_f(p,q)$ are obvious by some easy calculations:
Property 1 $\mathcal{H}_f(a,b;p,q)$ *are symmetric with respect to* a,b *and* p,q*, i.e.*

$$\mathcal{H}_f(a,b;p,q) = \mathcal{H}_f(a,b;q,p), \tag{2.1}$$

$$\mathcal{H}_f(a,b;p,q) = \mathcal{H}_f(b,a;p,q). \tag{2.2}$$

Property 2 Let

$$T(t) = \ln f(a^t,b^t). \tag{2.3}$$

Then

$$T'(t) = \frac{a^t f_x(a^t,b^t)\ln a + b^t f_y(a^t,b^t)\ln b}{f(a^t,b^t)} = \ln G_{f,t}(a,b), \tag{2.4}$$

where $t \neq 0$ if $(1,1) \notin \mathbb{U}$, $G_{f,t}(a,b)$ is defined by (1.8)
Property 3 *If* $T'(t)$ *is continuous on* $[p,q]$*, then*

$$\ln\mathcal{H}_f(p,q) = \frac{1}{p-q}\int_q^p T'(t)dt = \frac{1}{p-q}\int_q^p \ln G_{f,t}dt. \tag{2.5}$$

Property 4 Suppose $f(x,y)$ is a n-order homogeneous function for variable x and y,and $f(x,y) = f(y,x)$ for all $(x,y) \in \mathbb{U}(\subseteq \mathbb{R}_+ \times \mathbb{R}_+)$, then

$$f(a^{-t},b^{-t}) = G^{-2nt} f(a^t,b^t), \tag{2.6}$$

$$\mathcal{H}_f(t,-t) = G^n, \tag{2.7}$$

$$T(t) - T(-t) = 2nt\ln G, \tag{2.8}$$

where $G = \sqrt{ab}$.

The following Lemmas will be used in the proof of main results.

Lemma 1. *Let* $f(x,y)$ *be a positive* n*-order homogenous function defined on* $\mathbb{U}(\subseteq \mathbb{R}_+ \times \mathbb{R}_+)$ *and be 3-time differentiable. Then*

$$T'(t) = \tfrac{a^t f_x(a^t,b^t)\ln a + b^t f_y(a^t,b^t)\ln b}{f(a^t,b^t)} = \ln G_f^{\frac{1}{t}}(a^t,b^t), \tag{2.9}$$

$$T''(t) = -xyI_1\ln^2(b/a), \quad I_1 = (\ln f)_{xy}, \tag{2.10}$$

$$T'''(t) = -Ct^{-3}J, \quad J = (x-y)(xI_1)_x, \quad C = \tfrac{xy\ln^3(x/y)}{x-y} > 0, \tag{2.11}$$

in which $x = a^t, y = b^t$ *with* $t \neq 0$.

Remark 2. *By Lemma 1, it is no difficult to get the following conclusions:*
1) $T(t)$ *is strictly convex (concave) in* $t \in (-\infty,0)\cup(0,+\infty)$ *if* $I_1 < (>)0$;
2) $T'(t)$ *is strictly increasing (decreasing) in* $t \in (-\infty,0)\cup(0,+\infty)$ *if* $I_1 < (>)0$;
3) If $J < (>)0$*, then* $T'(t)$ *is strictly convex (concave) in* $t \in (0,+\infty)$*, and strictly concave (convex) in* $t \in (-\infty,0)$;
4) If $J < (>)0$*, then* $T''(t)$ *is strictly increasing (decreasing) in* $t \in (0,+\infty)$*, and strictly decreasing (increasing) in* $t \in (-\infty,0)$.

The following Lemma will be used in proof of Corollary 1 and 2.

Lemma 2. *The conditions of this Lemma are the same as in Lemma 1, and $f(x,y)$ is symmetric with respect to x and y, then the following equations hold:*

$$T'(t)+T'(-t) = 2n\ln G, \tag{2.12}$$
$$T''(-t) = T''(t), \tag{2.13}$$
$$T'''(-t) = -T'''(t), \tag{2.14}$$

where $t\neq 0, G=\sqrt{ab}$.

Proof. By direct calculations of the first, second and third derivative to variable t in two sides of equation (2.8) respectively, the equations (2.12)-(2.14) are derived immediately. The proof is completed. ∎

Remark 3. *If $(1,1)\in\mathbb{U}$, i.e. $T'(0)$ exists, then $T'(0)=n\ln G$; If $(1,1)\notin\mathbb{U}$, we define $T'(0)=\lim_{t\to 0}T'(t)=n\ln G$. Thus the (2.12) can be written as*

$$T'(t)+T'(-t)=2T'(0). \tag{2.15}$$

Corollary 4 is deduced from the following Lemma presented by Péter Czinder and Zsolt Páles (see[1]).

Lemma 3. *Let $f:\mathcal{J}\to R$ be symmetric with respect to an element $m\in\bar{\mathcal{J}}$, furthermore, suppose that f is convex over the interval $J\cap(-\infty,m]$ and concave over $J\cap[m,+\infty)$. Then, for any interval $[p,q]\subset\mathcal{J}$*

$$f(\frac{p+q}{2})\le(\ge)\frac{1}{p-q}\int_q^p f(t)dt\le(\ge)\frac{f(p)+f(q)}{2} \tag{2.16}$$

holds if $\frac{p+q}{2}\le(\ge)m$.

In (2.16) the reversed inequalities are valid if f is concave over the interval $J\cap[-\infty,m)$ and convex over $\mathcal{J}\cap[m,+\infty)$.

3. Proofs of Main Results

Proof of 2. It is sufficient to prove the convexity for p of $\ln\mathcal{H}_f$.

1) when $p\neq q$, $\ln\mathcal{H}_f=\dfrac{T(p)-T(q)}{p-q}$,

$$\frac{\partial\ln\mathcal{H}_f}{\partial p} = \frac{(p-q)T'(p)-T(p)+T(q)}{(p-q)^2}=\frac{g(p,q)}{(p-q)^2}, \tag{3.1}$$
$$\frac{\partial g(p,q)}{\partial p} = (p-q)T''(p) \tag{3.2}$$
$$\frac{\partial^2\ln\mathcal{H}_f}{\partial p^2} = \frac{(p-q)g_p(p,q)-2g(p,q)}{(p-q)^3}=\frac{k(p,q)}{(p-q)^3}, \tag{3.3}$$
$$\frac{\partial k(p,q)}{\partial p} = (p-q)^2T'''(p). \tag{3.4}$$

Notice $k(q,q)=0$. Obviously, if $T'''(p)>0$, then $\frac{\partial^2\ln\mathcal{H}_f}{\partial p^2}=\frac{k(p,q)}{(p-q)^3}>0$, i.e. $\ln\mathcal{H}_f$ is log-convexity in p; If $T'''(p)<0$, then it is reversed.

From Lemma 1, when $J=(x-y)(xI_1)_x<0$, if $p\in(0,+\infty)$,then $T'''(p)=-Cp^{-3}J>0$. While $p\in(-\infty,0)$, then $T'''(p)=-Cp^{-3}J<0$.

In the same way, when $J = (x - y)(xI_1)_x > 0$, if $p \in (0, +\infty)$,then $T'''(p) = -Cp^{-3}J < 0$. While $p \in (-\infty, 0)$, then $T'''(p) = -Cp^{-3}J > 0$.

2) when $p = q$. The proof was given by [4], of which details are omitted.

Combining 1) with 2), the proof is completed. ∎

proof of Corollary 3. It is enough to prove in the case of $J = (x-y)(xI_1)_x < 0$. 1) By Corollary 1, $\mathcal{H}_f(p, 1-p)$ is strictly monotone decreasing (increasing) in $p \in (0, \frac{1}{2})$, strictly monotone increasing (decreasing) in $p \in (\frac{1}{2}, 1)$.

2) If $p \in (1, +\infty)$ and $f(x, y)$ is symmetric with respect to x and y. Set

$$\alpha = \frac{p_2 - p_1}{p_2 - p_1 + 1}, \beta = \frac{1}{p_2 - p_1 + 1} \text{ with } 1 < p_1 < p_2, \tag{3.5}$$

then α , $\beta > 0$, $\alpha + \beta = 1$ and

$$\alpha p_2 + \beta(p_1 - 1) = p_2 - 1, \tag{3.6}$$

$$\alpha(p_1 - 1) + \beta p_2 = p_1. \tag{3.7}$$

By the log-convexity of $\mathcal{H}_f(p, q)$ in p on $(0, +\infty)$, we have

$$\begin{cases} \mathcal{H}_f^{\alpha}(p_2, 1-p_2)\mathcal{H}_f^{\beta}(p_1 - 1, 1 - p_2) > \mathcal{H}_f(p_2 - 1, 1 - p_2); \\ \mathcal{H}_f^{\alpha}(p_1 - 1, -p_1)\mathcal{H}_f^{\beta}(p_2, -p_1) > \mathcal{H}_f(p_1, -p_1). \end{cases} \tag{3.8}$$

Since $f(x, y) = f(y, x)$, it follows from (2.6) that

$$\begin{aligned} \mathcal{H}_f(p_1 - 1, 1 - p_2) &= \left[\frac{f(p_1 - 1)}{f(1 - p_2)}\right]^{\frac{1}{p_2 + p_1 - 2}} = G^{\frac{2n(p_1 - 1)}{p_2 + p_1 - 2}} \left[\frac{f(1 - p_1)}{f(1 - p_2)}\right]^{\frac{1}{p_2 + p_1 - 2}}, \\ \mathcal{H}_f(p_2 - 1, 1 - p_2) &= \mathcal{H}_f(p_1, -p_1) = G^n, \\ \mathcal{H}_f(p_1 - 1, -p_1) &= G^{2n}\mathcal{H}_f^{-1}(p_1, 1 - p_1), \\ \mathcal{H}_f(p_2, -p_1) &= \left[\frac{f(p_2)}{f(-p_1)}\right]^{\frac{1}{p_2 + p_1}} = G^{\frac{2np_1}{p_2 + p_1}} \left[\frac{f(p_2)}{f(p_1)}\right]^{\frac{1}{p_2 + p_1}}, \end{aligned}$$

and then (3.8) is equivalent to

$$\begin{cases} \mathcal{H}_f^{\alpha}(p_2, 1 - p_2)G^{\frac{2\beta n(p_1 - 1)}{p_2 + p_1 - 2}} \left[\frac{f(1 - p_1)}{f(1 - p_2)}\right]^{\frac{\beta}{p_2 + p_1 - 2}} > G^n, \\ G^{2\alpha n}\mathcal{H}_f^{-\alpha}(p_1, 1 - p_1)G^{\frac{2n\beta p_1}{p_2 + p_1}} \left[\frac{f(p_2)}{f(p_1)}\right]^{\frac{\beta}{p_2 + p_1}} > G^n. \end{cases} \tag{3.9}$$

Taking the $\frac{p_2 + p_1 - 2}{\beta}$-th, $\frac{p_2 + p_1}{\beta}$-th power of the two sides in the the above two inequalities, respectively, then

(3.10)

$$\begin{cases} \mathcal{H}_f^{\alpha(p_2 + p_1 - 2)}(p_2, 1 - p_2)G^{2\beta n(p_1 - 1)} \left[\frac{f(1 - p_1)}{f(1 - p_2)}\right]^{\beta} > G^{n(p_2 + p_1 - 2)}, \\ G^{2\alpha n(p_2 + p_1)}\mathcal{H}_f^{-\alpha(p_2 + p_1)}(p_1, 1 - p_1)G^{2n\beta p_1} \left[\frac{f(p_2)}{f(p_1)}\right]^{\beta} > G^{n(p_2 + p_1)}. \end{cases}$$

Let the left sides of two inequalities in (3.10) multiply each other and the right sides do also. Then we have

$$\begin{aligned} &\mathcal{H}_f^{\alpha(p_2 + p_1 - 2)}(p_2, 1 - p_2)\mathcal{H}_f^{-\alpha(p_2 + p_1)}(p_1, 1 - p_1) \left[\frac{f(1 - p_1)}{f(1 - p_2)}\frac{f(p_2)}{f(p_1)}\right]^{\beta} \\ &\qquad > G^{2n(p_2 + p_1 - 1)}G^{-2\beta n(2p_1 - 1) - 2\alpha n(p_2 + p_1)}, \end{aligned} \tag{3.11}$$

in which the left side equals to

$$\begin{aligned}
&\mathcal{H}_f^{\alpha(p_2+p_1-2)}(p_2,1-p_2)\mathcal{H}_f^{-\alpha(p_2+p_1)}(p_1,1-p_1)\left[\tfrac{f(1-p_1)}{f(p_1)}\tfrac{f(p_2)}{f(1-p_2)}\right]^{\beta}\\
=\ &\mathcal{H}_f^{\alpha(p_2+p_1-2)+\beta(2p_2-1)}(p_2,1-p_2)\mathcal{H}_f^{-\alpha(p_2+p_1)+\beta(1-2p_1)}(p_1,1-p_1)\\
=\ &\mathcal{H}_f^{p_2+p_1-1}(p_2,1-p_2)\mathcal{H}_f^{-(p_2+p_1-1)}(p_1,1-p_1),
\end{aligned}$$

the right side equals to 1, because

$$\begin{aligned}
&2n(p_2+p_1-1)-2\beta n(2p_1-1)-2\alpha n(p_2+p_1)\\
=\ &2n(p_2+p_1-1)-\frac{2n(2p_1-1)+2n(p_2+p_1)(p_2-p_1)}{p_2-p_1+1}\\
=\ &2n[(p_2+p_1-1)-\frac{(2p_1-1)+(p_2+p_1)(p_2-p_1)}{p_2-p_1+1}\\
=\ &2n\left[(p_2+p_1-1)-\frac{p_2^2-(p_1^2-2p_1+1)}{p_2-p_1+1}\right]\\
=\ &0.
\end{aligned}$$

Consequently, there is

$$\mathcal{H}_f^{p_2+p_1-1}(p_2,1-p_2)\mathcal{H}_f^{-(p_2+p_1-1)}(p_1,1-p_1)>1$$

from (3.11), which is equivalent to

$$\mathcal{H}_f(p_2,1-p_2)>\mathcal{H}_f(p_1,1-p_1) \tag{3.12}$$

for $p_2+p_1-1>0$, i.e. $\mathcal{H}_f(p,1-p)$ is strictly increasing in p on $(1,+\infty)$ if $f(x,y)$ is symmetric with respect to x and y.

If $p\in(-\infty,0)$. Assume $p_1,p_2\in(-\infty,0)$ with $p_1<p_2$, then $1-p_2,1-p_1\in(1,+\infty)$ with $1-p_2<1-p_1$. It follows from (3.12) that

$$\mathcal{H}_f(1-p_1,1-(1-p_1))>\mathcal{H}_f(1-p_2,1-(1-p_2)), \tag{3.13}$$

i.e.

$$\mathcal{H}_f(1-p_1,p_1)>\mathcal{H}_f(1-p_2,p_2). \tag{3.14}$$

By (2.2), inequality (3.12) is reversed, which shows that $\mathcal{H}_f(p,1-p)$ is strictly decreasing in p on $(-\infty,0)$.

Combining 1) with 2), the proof is completed. ∎

proof of Corollary 4. It proves only in the case of $J=(x-y)(xI_1)_x<0$.

Since $f(x,y)$ is defined on $\mathbb{R}_+\times\mathbb{R}_+$ and is symmetric with respect to x and y further, by (2.15), $T'(t):\mathbb{R}_+\to\mathbb{R}_+$ is symmetric with respect to 0; It follows from (2.11) that $T'(t)$ is strictly convex in t on $(0,+\infty)$ and concave on $(-\infty,0)$ if $J=(x-y)(xI_1)_x<0$.

Using Lemma 3, that

$$T'(\frac{p+q}{2})<(>)\frac{1}{p-q}\int_q^p T'(t)\mathrm{d}t<(>)\frac{T'(p)+T'(q)}{2} \tag{3.15}$$

holds if $\frac{p+q}{2}>(<)0$ with $p\neq q$.

It follows from (2.5) that (1.10) is true.

The proof is completed. ∎

4. Refinements of Some Conclusion about L, A, E and D

Applying Theorem 2 and Corollary 3 and 4, the Conclusions about L, A, E and D in [4] can be refined and extended by:

Conclusion 1. *For $f(x,y) = L(x,y)$, $A(x,y)$ and $E(x,y)$,*

1) $\mathcal{H}_f(p,q)$ are strictly log-concave with respect to either p or q on $(0,+\infty)$, and strictly log-convex on $(-\infty,0)$.

2) $\mathcal{H}_f(p,1-p)$ are strictly increasing in p on $(-\infty,\frac{1}{2})$, and strictly decreasing on $(\frac{1}{2},+\infty)$.

3) If $p+q>0$, then

$$G_{f,\frac{p+q}{2}} > \mathcal{H}_f(p,q) > \sqrt{G_{f,p}G_{f,q}}. \tag{4.1}$$

Inequality (4.1) is reversed if $p+q<0$.

Conclusion 2. *1) $\mathcal{H}_D(p,q)$ is strictly log-convex with respect to either p or q on $(0,+\infty)$, and strictly log-concave on $(-\infty,0)$.*

2) $\mathcal{H}_D(p,1-p)$ is strictly decreasing in p on $(-\infty,0)\cup(0,\frac{1}{2})$, and strictly increasing on $(\frac{1}{2},1)\cup(1,+\infty)$.

Using Conclusion 1, the (3.8), (3.9) and (3.10) in [4] can be extended by

$$\begin{aligned} G^{\frac{2}{3}}A^{\frac{1}{3}} &< \sqrt{Gh_1} < G^{\frac{2}{5}}M_{\frac{1}{3}}^{\frac{1}{5}}M_{\frac{2}{3}}^{\frac{2}{5}} < L < M_{\frac{1}{5}}^{\frac{1}{3}}M_{\frac{2}{5}}^{\frac{2}{3}} \\ &< h_{\frac{1}{2}} < M_{\frac{1}{3}} < h_{\frac{2}{5}}^2 M_{\frac{1}{5}}^{-1} < E_{\frac{1}{2}}, \end{aligned} \tag{4.2}$$

where $M_p = (\dfrac{a^p+b^p}{2})^{\frac{1}{p}}, E_p = E^{\frac{1}{p}}(a^p,b^p), E(a,b) = e^{-1}\left(\frac{b^b}{a^a}\right)^{\frac{1}{b-a}}, h_p = \left[\dfrac{a^p+(\sqrt{ab})^p+b^p}{3}\right]^{\frac{1}{p}}$.

$$\begin{aligned} G^{\frac{2}{3}}M_2^{\frac{2}{3}}A^{-\frac{1}{3}} &< G^{\frac{1}{2}}M_{\frac{3}{2}}^{\frac{3}{4}}M_{\frac{1}{2}}^{-\frac{1}{4}} < G^{\frac{2}{5}}M_{\frac{4}{3}}^{\frac{4}{5}}M_{\frac{1}{3}}^{-\frac{1}{5}} < A < M_{\frac{4}{5}}^{\frac{4}{3}}M_{\frac{1}{5}}^{-\frac{1}{3}} \\ &< M_{\frac{3}{4}}^{\frac{3}{2}}M_{\frac{1}{4}}^{-\frac{1}{2}} < M_{\frac{2}{3}}^{2}M_{\frac{1}{3}}^{-1} < M_{\frac{3}{5}}^{3}M_{\frac{2}{5}}^{-2} < Z_{\frac{1}{2}}, \end{aligned} \tag{4.3}$$

where $M_p = (\dfrac{a^p+b^p}{2})^{\frac{1}{p}}$, $Z_p = Z^{\frac{1}{p}}(a^p,b^p), Z(a,b) = a^{\frac{a}{a+b}}b^{\frac{b}{a+b}}$.

$$\begin{aligned} G^{\frac{2}{3}}Z_1^{\frac{1}{3}} &< G^{\frac{1}{2}}E_{\frac{3}{2}}^{\frac{3}{4}}E_{\frac{1}{2}}^{-\frac{1}{4}} < G^{\frac{2}{5}}Z_{\frac{1}{3}}^{\frac{1}{5}}Z_{\frac{2}{3}}^{\frac{2}{5}} < E < Z_{\frac{1}{5}}^{\frac{1}{3}}Z_{\frac{2}{5}}^{\frac{2}{3}} \\ &< E_{\frac{3}{4}}^{\frac{3}{2}}E_{\frac{1}{4}}^{-\frac{1}{2}} < Z_{\frac{1}{3}} < E_{\frac{3}{5}}^{3}E_{\frac{2}{5}}^{-2} < Y_{\frac{1}{2}}, \end{aligned} \tag{4.4}$$

where $Z_p = Z^{\frac{1}{p}}(a^p,b^p)$, $E_p = E^{\frac{1}{p}}(a^p,b^p)$, $Y_p = Y^{\frac{1}{p}}(a^p,b^p), Y(a,b) = Ee^{1-\frac{G^2}{L^2}}$.

If replace a, b with a^2, b^2 in (4.2)-(4.4), then they may be rewritten into

$$\begin{aligned} &E > h^2_{\frac{4}{5}} M^{-1}_{\frac{2}{5}} > M_{\frac{2}{3}} > h > M^{\frac{1}{3}}_{\frac{2}{5}} M^{\frac{2}{3}}_{\frac{4}{5}} > \\ &L_2 > G^{\frac{2}{5}} M^{\frac{1}{5}}_{\frac{2}{3}} M^{\frac{2}{5}}_{\frac{4}{3}} > \sqrt{G h_2} > G^{\frac{2}{3}} M^{\frac{1}{3}}_2, \end{aligned} \tag{4.5}$$

$$\begin{aligned} &Z > M^3_{\frac{6}{5}} M^{-2}_{\frac{4}{5}} > M^2_{\frac{4}{3}} M^{-1}_{\frac{2}{3}} > M^{\frac{3}{2}}_{\frac{3}{2}} M^{-\frac{1}{2}}_{\frac{1}{2}} > M^{\frac{4}{3}}_{\frac{8}{5}} M^{-\frac{1}{3}}_{\frac{2}{5}} > \\ &M_2 > G^{\frac{2}{5}} M^{\frac{4}{5}}_{\frac{8}{3}} M^{-\frac{1}{5}}_{\frac{2}{3}} > G^{\frac{1}{2}} M^{\frac{3}{4}}_3 A^{-\frac{1}{4}} > G^{\frac{2}{3}} M^{\frac{2}{3}}_4 M^{-\frac{1}{3}}_2, \end{aligned} \tag{4.6}$$

$$\begin{aligned} &Y > E^3_{\frac{6}{5}} E^{-2}_{\frac{4}{5}} > Z_{\frac{2}{3}} > E^{\frac{3}{2}}_{\frac{3}{2}} E^{-\frac{1}{2}}_{\frac{1}{2}} > Z^{\frac{1}{3}}_{\frac{2}{5}} Z^{\frac{2}{3}}_{\frac{4}{5}} > \\ &E_2 > G^{\frac{2}{5}} Z^{\frac{1}{5}}_{\frac{2}{3}} Z^{\frac{2}{5}}_{\frac{4}{3}} > G^{\frac{1}{2}} E^{\frac{3}{4}}_3 E^{-\frac{1}{4}}_1 > G^{\frac{2}{3}} Z^{\frac{1}{3}}_2. \end{aligned} \tag{4.7}$$

respectively, where $L_p = L^{\frac{1}{p}}(a^p, b^p)$.

Form (4.7) it follows that

$$Y > Z_{\frac{2}{3}} > E_2; \tag{4.8}$$

Likewise from (4.5), we also have

$$E > M_{\frac{2}{3}} > L_2. \tag{4.9}$$

And then we can get a new inequalities chain:

$$Y > Z_{\frac{2}{3}} > E_2 > M_{\frac{4}{3}} > L_4, \tag{4.10}$$

which precisely characterize the relations among means Y, Z, E, A and L, and is very interesting.

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