THE WEIGHTED HERON MEAN OF SEVERAL POSITIVE NUMBERS

ZHEN-GANG XIAO, ZHI-HUA ZHANG, AND V. LOKESHA

Abstract. In this paper, a definition of the weighted Heron mean of several positive numbers is given, its monotonicity is proved, and an identity relating to the same mean is obtained.

1. Introduction

For positive numbers a_0, a_1, let

$$L = L(a_0, a_1) = \begin{cases} \dfrac{a_0 - a_1}{\ln a_0 - \ln a_1}, & a_0 \neq a_1; \\ a_0, & a_0 = a_1; \end{cases} \tag{1.1}$$

$$H = H(a_0, a_1) = \frac{a_0 + \sqrt{a_0 a_1} + a_1}{3}. \tag{1.2}$$

These are respectively called the logarithmic and Heron means (see [1]).

In 2004, Zhang and Wu [2] gave the generalization of Heron mean and its dual form in two variables respectively as follows

$$H(a_0, a_1; k) = \frac{1}{k+1} \sum_{i=0}^{k} a_0^{\frac{k-i}{k}} a_1^{\frac{i}{k}}, \tag{1.3}$$

and

$$h(a_0, a_1; k) = \frac{1}{k} \sum_{i=1}^{k} a_0^{\frac{k+1-i}{k+1}} a_1^{\frac{i}{k+1}}, \tag{1.4}$$

where k is a natural number. Authors proved that $H(a_0, a_1; k)$ is a monotone decreasing function and $h(a_0, a_1; k)$ is a monotone increasing function with k, and

$$\lim_{k\to+\infty} H(a_0, a_1; k) = \lim_{k\to+\infty} h(a_0, a_1; k) = L(a_0, a_1).$$

Let $a = (a_0, a_1, \cdots, a_n)$ and r be a nonnegative integer, where a_i for $0 \leq i \leq n$ are nonnegative real numbers. Then

$$H_n^{[r]} = H_n^{[r]}(a) = \frac{1}{\binom{n+r}{r}} \sum_{\substack{i_0+i_1+\cdots+i_n=r, \\ i_0, i_1, \cdots, i_n \geq 0 \text{ are integers}}} \prod_{k=0}^{n} a_k^{i_k/r} \tag{1.5}$$

is called the generalized Heron mean of a.

In 2003, Xiao and Zhang [3] obtained that for any nonnegative integers r, s with $s > r$, then

$$H_n^{[r]}(a) \geqslant H_n^{[s]}(a), \tag{1.6}$$

with the equality if and only if $a_0 = a_1 = \cdots = a_n$, and

$$\lim_{r\to\infty} H_n^{[r]}(a) = L(a) = \frac{n! V(\ln a; 1, 0)}{V(\ln a; 0, n)}, \tag{1.7}$$

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where $L(a)$ is called the logarithmic mean in n variables, and $\ln a = (\ln a_0, \ln a_1, \ldots, \ln a_n)$, $a_i \neq a_j$ for $i \neq j$,

$$V(\ln a; r, k) = \begin{vmatrix} 1 & \ln a_0 & \ln^2 a_0 & \cdots & \ln^{n-1} a_0 & a_0^r \ln^k a_0 \\ 1 & \ln a_1 & \ln^2 a_1 & \cdots & \ln^{n-1} a_1 & a_1^r \ln^k a_1 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & \ln a_n & \ln^2 a_n & \cdots & \ln^{n-1} a_n & a_n^r \ln^k a_n \end{vmatrix}. \tag{1.8}$$

In this paper, a definition of the weighted Heron mean of several positive numbers is given, its monotonicity is proved, and an identity relating to it is obtained.

2. Main Results

Definition 2.1. *Let $a = (a_0, a_1, \cdots, a_n), \lambda = (\lambda_0, \lambda_1, \cdots, \lambda_n)$ with $a_i \geqslant 0$ and $\lambda_i > 0$ for $0 \le i \le n$, and r be a nonnegative integer. Then*

$$H_n^{[r]}(a, \lambda) = \frac{1}{\binom{n+r+1}{r} \sum_{i=0}^n \lambda_i} \sum_{\substack{i_0+i_1+\cdots+i_n=r, \\ i_0, i_1, \cdots, i_n \geq 0 \text{ are integers}}} \left(\sum_{k=0}^n (1+i_k)\lambda_k \right) \prod_{k=0}^n a_k^{i_k/r} \tag{2.1}$$

is called the weighted Heron mean of a for λ.

Now, we give some theorems relating to the weighted Heron mean $H_n^{[r]}(a)$, the proof of Theorem 2.1 is left to next section.

Theorem 2.1. *If $r \in \mathbb{N}$, then*

$$H_n^{[r]}(a, \lambda) = \frac{\int_E (\sum_{i=0}^n \lambda_i x_i)(\sum_{i=0}^n a_i^{1/r} x_i)^r dx}{\int_E (\sum_{i=0}^n \lambda_i x_i) dx} \tag{2.2}$$

where $x_0 = 1 - \sum_{i=1}^n x_i$, and $dx = dx_1 dx_2 \cdots dx_n$ denotes the differential of the volume in E :

$$E = \left\{ (x_1, x_2, \cdots, x_n) : \sum_{i=1}^n x_i \leqslant 1, x_i \geqslant 0, i = 1, 2, \cdots, n \right\}. \tag{2.3}$$

Theorem 2.2. *If $r \in \mathbb{N}$, then $H_n^{[r]}(a, \lambda)$ is a monotone decreasing function with r, that is*

$$H_n^{[r]}(a, \lambda) \leqslant H_n^{[r+1]}(a, \lambda), \tag{2.4}$$

with equality holding if and only if $a_0 = a_1 = \cdots = a_n$.

Proof. From well-known power mean inequality, we have that

$$M_r(a, x) = \begin{cases} \left(\dfrac{\sum_{k=0}^n a_k^r x_k}{\sum_{k=0}^n x_k} \right)^{\frac{1}{r}}, & r \neq 0; \\ \prod\limits_{k=0}^n a_k^{x_k / \sum_{k=0}^n x_k}, & r = 0; \end{cases} \tag{2.5}$$

is a monotone increasing function with r, or $M_{\frac{1}{r}}(a, x)$ is a monotone decreasing function with r.

Combining expression (2.2) and (2.5), we immediately obtain that $H_n^{[r]}(a, \lambda)$ is a monotone decreasing function with $r \in \mathbb{N}$. The proof of Theorem 2.2 is completed. □

Theorem 2.3. *If $r \in \mathbb{N}$, then*

$$\lim_{r \to \infty} H_n^{[r]}(a, \lambda) = \frac{\int_E (\sum_{i=0}^n \lambda_i x_i)(\prod_{i=0}^n a_i^{x_i}) dx}{\int_E (\sum_{i=0}^n \lambda_i x_i) dx}, \tag{2.6}$$

where x_0, dx and E denote as Theorem 2.1.

Proof. This follows from (2.5), Theorem 2.1 and standard arguments. □

3. The Proof of Theorem 2.1

Throughout this section we assume $\mathbb{R}$ is a set of real numbers, and $\mathbb{R}_+$ is a set of strictly positive real numbers. Let $a = (a_0, a_1, \cdots, a_n) \in \mathbb{R}_+^{n+1}$, $a_i \neq a_j$ for $i \neq j$, and φ be a function in $\mathbb{R}$, taking

$$V(a;\varphi) = \begin{vmatrix} 1 & a_0 & a_0^2 & \cdots & a_0^{n-1} & \varphi(a_0) \\ 1 & a_1 & a_1^2 & \cdots & a_1^{n-1} & \varphi(a_1) \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & a_n & a_n^2 & \cdots & a_n^{n-1} & \varphi(a_n) \end{vmatrix}. \tag{3.1}$$

Let $\varphi(x) = x^{n+r} \ln^k x$, then we have

$$V(a;r,k) = \begin{vmatrix} 1 & a_0 & a_0^2 & \cdots & a_0^{n-1} & a_0^{n+r}\ln^k a_0 \\ 1 & a_1 & a_1^2 & \cdots & a_1^{n-1} & a_1^{n+r}\ln^k a_1 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & a_n & a_n^2 & \cdots & a_n^{n-1} & a_n^{n+r}\ln^k a_n \end{vmatrix}. \tag{3.2}$$

Note the case $r = 0$ and $k = 0$ is just the determinant of Van der Monde's matrix of order $n+1$:

$$V(a;0,0) = \sum_{i=0}^{n} (-1)^{n+i} a_i^n V_i(a) = \prod_{0 \le i < j \le n} (a_j - a_i), \tag{3.3}$$

where

$$V_i(a) = \begin{vmatrix} 1 & a_0 & a_0^2 & \cdots & a_0^{n-1} \\ 1 & a_1 & a_1^2 & \cdots & a_1^{n-1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & a_{i-1} & a_{i-1}^2 & \cdots & a_{i-1}^{n-1} \\ 1 & a_{i+1} & a_{i+1}^2 & \cdots & a_{i+1}^{n-1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & a_n & a_n^2 & \cdots & a_n^{n-1} \end{vmatrix}, \quad (0 \le i \le n). \tag{3.4}$$

Also let $0 \leqslant i \leqslant n$, we set

$$V(a,i;\varphi) = \begin{vmatrix} 1 & a_0 & a_0^2 & \cdots & a_0^{n-1} & \varphi(a_0) \\ 1 & a_1 & a_1^2 & \cdots & a_1^{n-1} & \varphi(a_1) \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & a_i & a_i^2 & \cdots & a_i^{n-1} & \varphi(a_i) \\ 0 & 1 & 2a_i & \cdots & (n-1)a_i^{n-2} & \varphi'(a_i) \\ 1 & a_{i+1} & a_{i+1}^2 & \cdots & a_{i+1}^{n-1} & \varphi(a_{i+1}) \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & a_n & a_n^2 & \cdots & a_n^{n-1} & \varphi(a_n) \end{vmatrix}, \tag{3.5}$$

and for $\varphi(x) = x^{n+r+1}$ in (3.5), we have

$$V(a,i;r) = \begin{vmatrix} 1 & a_0 & a_0^2 & \cdots & a_0^{n} & a_0^{n+r+1} \\ 1 & a_1 & a_1^2 & \cdots & a_1^{n} & a_1^{n+r+1} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & a_i & a_i^2 & \cdots & a_i^{n} & a_i^{n+r+1} \\ 0 & 1 & 2a_i & \cdots & na_i^{n-1} & (n+r+1)a_i^{n+r} \\ 1 & a_{i+1} & a_{i+1}^2 & \cdots & a_{i+1}^{n} & a_{i+1}^{n+r+1} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & a_n & a_n^2 & \cdots & a_n^{n} & a_n^{n+r+1} \end{vmatrix} \tag{3.6}$$

for $i \leqslant i \leqslant n$, and

$$V(a,i;0) = (-1)^{i+1} V(a;0,0) \prod_{j=0, j\neq i}^{n} (a_j - a_i) = (-1)^{i+1} V^2(a;0,0)/V_i(a). \tag{3.7}$$

Lemma 3.1. *If $n \in N$, φ be a $(n+1)$-orders differentiable function on interval $\mathbb{I} \subset \mathbb{R}_+$, then we have*

$$V(a;\varphi) = V(a;0,0)\int_E \varphi^{(n)}(A(a,x))dx, \tag{3.8}$$

and

$$\sum_{i=0}^{n}(-1)^{i+1}\lambda_i V(a,i;\varphi)V_i(a) = V^2(a;0,0)\int_E A(\lambda,x)\varphi^{(n+1)}(A(a,x))dx, \tag{3.9}$$

where dx, E denote as Theorem 2.1, and $a_i, \in \mathbb{I}$, $A(a,x) = a_0 + \sum_{i=1}^{n}(a_i - a_0)x_i = \sum_{i=0}^{n} a_i x_i$, $x_0 = 1 - \sum_{i=1}^{n} x_i$.

Proof. Expression (3.8) is obtained in [4] and [5].

We prove expression (3.9). It is easy to know that

$$\begin{aligned}
V(a,i;r) &\\
&= \sum_{k=0}^{i-1}(-1)^{n+k+i}\varphi(a_k)\cdot V_k(a)\prod_{j=0,j\neq i,k}^{n}(a_j - a_i)\\
&+ \varphi(a_i)\cdot\left[\sum_{k=0}^{i-1}(-1)^{n+k+i+1}V_k(a)\prod_{j=0,j\neq i,k}^{n}(a_j-a_i) + \sum_{k=i+1}^{n}(-1)^{n+k+i}V_k(a)\prod_{j=0,j\neq i,k}^{n}(a_j-a_i)\right]\\
&+ \sum_{k=i+1}^{n}(-1)^{n+k+i+1}\varphi(a_k)\cdot V_k(a)\prod_{j=0,j\neq i,k}^{n}(a_j-a_i) + (-1)^{n+i}\varphi'(a_i)\cdot V(a;0,0).
\end{aligned} \tag{3.10}$$

Let $A_0 = \int_E \left(1 - \sum_{k=1}^{n} x_k\right)\varphi^{(n+1)}(A(a,x))dx$, $A_i = \int_E x_i\varphi^{(n)}(A(a,x))dx$ $(1 \leqslant i \leqslant n)$, then

(3.11)

$$\begin{aligned}
\int_E A(\lambda,x)\varphi^{(n+1)}(A(a,x))dx &= \int_E\left(\sum_{i=0}^{n}\lambda_i x_i\right)\varphi^{(n+1)}(A(a,x))dx\\
&= \lambda_0\int_E\left(1-\sum_{k=1}^{n}x_k\right)\varphi^{(n+1)}(A(a,x))dx + \sum_{i=1}^{n}\lambda_i\int_E x_i\varphi^{(n+1)}(A(a,x))dx\\
&= \lambda_0 A_0 + \sum_{i=1}^{n}\lambda_i A_i.
\end{aligned}$$

If $x_0 = 1 - \sum_{i=1}^{n} x_i$, then $x_n = 1 - \sum_{i=0}^{n-1} x_i$, and we have

$$\begin{aligned}
E &= \left\{(x_1,x_2,\cdots,x_n) : \sum_{i=1}^{n} x_i \leqslant 1, x_i \geqslant 0, i = 1,2,\cdots,n\right\},\\
F &= \left\{(x_0,x_1,\cdots,x_{n-1}) : \sum_{i=0}^{n-1} x_i \leqslant 1, x_i \geqslant 0, i = 0,1,\cdots,n-1\right\},\\
\varphi^{(n+1)}(A(a,x)) &= \varphi^{(n+1)}\left(a_0 + \sum_{i=1}^{n}(a_i-a_0)x_i\right) = \varphi^{(n+1)}\left(a_n + \sum_{i=0}^{n-1}(a_i-a_n)x_i\right).
\end{aligned}$$

Therefore

$$A_0 = \int_E\left(1-\sum_{k=1}^{n}x_k\right)\varphi^{(n+1)}(A(a,x))dx = \int_F x_0\varphi^{(n+1)}(A(a,x))dx^*,$$

where $dx^* = dx_0dx_1\cdots dx_{n-1}$, that is still the form of $A_i = \int_E x_i\varphi^{(n+1)}(x)dx$ for $1 \leqslant i \leqslant n$.

Let $\bar{a}_i = (a_0, \cdots, a_{i-1}, a_{i+1}, \cdots, a_n)$, then

$$V(a,i;0) = (-1)^{i+1} V(a;0,0) \prod_{j=0, j\neq i}^{n} (a_j - a_i) \tag{3.12}$$

$$= V(\bar{a}_i;0,0) \prod_{j=0, j\neq i}^{n} (a_j - a_i)^2 V_k(a)$$

$$= \begin{cases} (-1)^i V_k(\bar{a}_i) \displaystyle\prod_{j=0, j\neq i,k}^{n} (a_j - a_i), & (0 \leqslant k < i), \\ (-1)^{i+1} V_{k-1}(\bar{a}_i) \displaystyle\prod_{j=0, j\neq i,k}^{n} (a_j - a_i), & (i < k \leqslant n); \end{cases}$$

$$\sum_{k=0, k\neq i}^{n} (-1)^{n+k} V_k(a) = (-1)^{n+i-1} V_i(a). \tag{3.13}$$

From expression (3.8), we obtain

$$V(\bar{a};0,0) \int_0^{1-x_i} \int_0^{1-x_i-x_1} \cdots \int_0^{1-\sum_{i=1}^{n-1} x_i} \varphi^{(n+1)}(A(a,x)) dx_1 dx_2 \cdots dx_n \tag{3.14}$$
$$= \sum_{k=0}^{i-1} (-1)^{n+k} V_k(\bar{a}_k) \varphi''(a_k + (a_i - a_k)x_i) + \sum_{k=i+1}^{n} (-1)^{n+k-1} V_{k-1}(\bar{a}_k) \varphi''(a_k + (a_i - a_k)x_i),$$

and

$$(a_k - a_i)^2 \int_0^1 x_i \varphi''(a_k + (a_i - a_k)x_i) dx_i = \varphi(a_k) - \varphi(a_i) - (a_k - a_i)\varphi'(a_i). \tag{3.15}$$

Hence

$$V(a,i;0)A_i$$
$$= (-1)^{i+1} V(a;0,0) \prod_{j=0, j\neq i}^{n} (a_j - a_i) \int_E x_i \varphi^{(n+1)}(A(a,x)) dx$$
$$= V(\bar{a}_i;0,0) \prod_{j=0, j\neq i}^{n} (a_j - a_i)^2 \int_0^1 \int_0^{1-x_1} \int_0^{1-\sum_{i=1}^{n-1} x_i} x_i \varphi^{(n+1)}(A(a,x)) dx_1 dx_2 \cdots dx_n$$
$$= \prod_{j=0, j\neq i}^{n} (a_j - a_i)^2 \int_0^1 x_i \left[V(\bar{a}_i;0,0) \int_0^{1-x_i} \int_0^{1-x_i-x_1} \cdots \int_0^{1-\sum_{i=1}^{n-1} x_i} \varphi^{(n+1)}(A(a,x)) dx_1 \cdots dx_n \right] dx_i$$
$$= \sum_{k=0}^{i-1} (-1)^{n+k} V_k(\bar{a}_i) \prod_{j=0, j\neq i,k}^{n} (a_j - a_i)^2 [\varphi(a_k) - \varphi(a_i) - (a_k - a_i)\varphi'(a_i)]$$
$$+ \sum_{k=i+1}^{n} (-1)^{n+k-1} V_{k-1}(\bar{a}_i) \prod_{j=0, j\neq i,k}^{n} (a_j - a_i)^2 [\varphi(a_k) - \varphi(a_i) - (a_k - a_i)\varphi'(a_i)]$$
$$= \sum_{k=1}^{i-1} (-1)^{n+k+i} \varphi(a_k) \cdot V_k(a) \prod_{j=0, j\neq i,k}^{n} (a_j - a_i) + \varphi(a_i) \cdot \left[\sum_{k=0}^{i-1} (-1)^{n+k+i+1} V_k(a) \prod_{j=0, j\neq i,k}^{n} (a_j - a_i) \right.$$
$$\left. + \sum_{k=i+1}^{n} (-1)^{n+k+i} V_k(a) \prod_{j=0, j\neq i,k}^{n} (a_j - a_i) \right] + \sum_{k=i+1}^{n} (-1)^{n+k+i+1} \varphi(a_k) \cdot V_k(a) \prod_{j=0, j\neq i,k}^{n} (a_j - a_i)$$
$$+ (-1)^{n+i} V(a;0,0) \cdot \varphi'(a_i)$$
$$= V(a,i;\varphi),$$

i.e.

$$A_i = \int_E x_i \varphi^{(n+1)}(A(a,x))dx = \frac{V(a,i;\varphi)}{V(a,i;0)} = (-1)^{i+1} \frac{V(a,i;\varphi)\cdot V_i(a)}{V^2(a;0,0)}. \tag{3.16}$$

Combining (3.11) and (3.16), we find expression (3.9). The proof of Lemma 3.1 is completed. □

Lemma 3.2. *Let r be an integer, then*

$$V(a;r,0) = \begin{cases} V(a;0,0)\cdot \displaystyle\sum_{\substack{i_0+i_1+\cdots+i_n=r,\\ i_0,i_1,\cdots,i_n\geq 0 \text{ are integers}}} \prod_{k=0}^{n} a_k^{i_k}, & r \geqslant 0; \\ 0, & r=-1,-2,\cdots,-n; \\ (-1)^n V(a;0,0)\cdot \displaystyle\sum_{\substack{i_0+i_1+\cdots+i_n=-r,\\ i_0,i_1,\cdots,i_n\geq 1 \text{ are integers}}} \prod_{k=0}^{n} a_k^{-i_k}, & r<-n. \end{cases} \tag{3.17}$$

Proof. Taking $V_n(a;r,0) := V(a;r,0)$. Obviously, if $r=-1,-2,\cdots,-n$, then $V_n(a;r,0) = V(a;r,0) = 0$.

For $n \in \mathbb{N}$, $r \geqslant 0$, It will be verified by mathematical induction. It is clear that identity (3.17) holds trivially for $n=1$ and $r \geqslant 0$, since

$$\begin{aligned} V_2(a;r,0) &= a_1^{r+1} - a_0^{r+1} \\ &= (a_1-a_0)\left(a_1^r + a_1^{r-1}a_0 + \cdots + a_0^r\right) \\ &= (a_1-a_0) \sum_{\substack{i_0+i_1=r,\\ i_0,i_1\geq 0 \text{ are integers}}} a_0^{i_0}a_1^{i_1}. \end{aligned}$$

Suppose identity (3.17) is true for $n-1$ and integers $t \geqslant 0$. That is

$$V_{n-1}(a;t,0) = V_{n-1}(a;0,0)\cdot \sum_{\substack{i_0+i_1+\cdots+i_{n-1}=t,\\ i_0,i_1,\cdots,i_{n-1}\geq 0 \text{ are integers}}} \prod_{k=0}^{n-1} a_k^{i_k}. \tag{3.18}$$

By (3.2), from (3.18), we have

$$\begin{aligned} V_n(a;r,0) &= \begin{vmatrix} 1 & a_0-a_n & a_0^2-a_0a_n & \cdots & a_0^{n-1}-a_0^{n-2}a_n & a_0^{n+r}-a_0^{n-1}a_n^{r+1} \\ 1 & a_1-a_n & a_1^2-a_1a_n & \cdots & a_1^{n-1}-a_1^{n-2}a_n & a_1^{n+r}-a_1^{n-1}a_n^{r+1} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & a_{n-1}-a_n & a_{n-1}^2-a_{n-1}a_n & \cdots & a_{n-1}^{n-1}-a_{n-1}^{n-2}a_n & a_{n-1}^{n+r}-a_{n-1}^{n-1}a_n^{r+1} \\ 1 & 0 & 0 & \cdots & 0 & 0 \end{vmatrix} \\ &= (-1)^{n+2} \begin{vmatrix} a_0-a_n & a_0^2-a_0a_n & \cdots & a_0^{n-1}-a_0^{n-2}a_n & a_0^{n+r}-a_0^{n-1}a_n^{r+1} \\ a_1-a_n & a_1^2-a_1a_n & \cdots & a_1^{n-1}-a_1^{n-2}a_n & a_1^{n+r}-a_1^{n-1}a_n^{r+1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n-1}-a_n & a_{n-1}^2-a_{n-1}a_n & \cdots & a_{n-1}^{n-1}-a_{n-1}^{n-2}a_n & a_{n-1}^{n+r}-a_{n-1}^{n-1}a_n^{r+1} \end{vmatrix} \\ &= \prod_{i=0}^{n-1}(a_n-a_i) \begin{vmatrix} 1 & a_0 & a_0^2 & \cdots & a_0^{n-1} & \sum_{t+i_n=r} a_0^{n-1+t}a_n^{i_n} \\ 1 & a_1 & a_1^2 & \cdots & a_1^{n-1} & \sum_{t+i_n=r} a_1^{n-1+t}a_n^{i_n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & a_{n-1} & a_{n-1}^2 & \cdots & a_{n-1}^{n-1} & \sum_{t+i_n=r} a_{n-1}^{n-1+t}a_n^{i_n} \end{vmatrix} \\ &= \prod_{i=0}^{n-1}(a_n-a_i) \sum_{t+i_n=r} a_n^{i_n} \begin{vmatrix} 1 & a_0 & a_0^2 & \cdots & a_0^{n-1} & a_0^{n-1+t} \\ 1 & a_1 & a_1^2 & \cdots & a_1^{n-1} & a_1^{n-1+t} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & a_{n-1} & a_{n-1}^2 & \cdots & a_{n-1}^{n-1} & a_{n-1}^{n-1+t} \end{vmatrix} \end{aligned}$$

$$
\begin{aligned}
&= \prod_{i=0}^{n-1}(a_n - a_i) \sum_{t+i_n=r} a_n^{i_n} V_{n-1}(a;t,0) \\
&= \prod_{i=0}^{n-1}(a_n - a_i) \cdot V_{n-1}(a;0,0) \sum_{t+i_n=r} a_n^{i_n} \sum_{\substack{i_0+i_1+\cdots+i_{n-1}=t,\\ i_0,i_1,\cdots,i_{n-1}\geq 0 \text{ are integers}}} \prod_{k=0}^{n-1} a_k^{i_k} \\
&= V(a;0,0) \cdot \sum_{\substack{i_0+i_1+\cdots+i_n=r,\\ i_0,i_1,\cdots,i_n\geq 0 \text{ are integers}}} \prod_{k=0}^{n} a_k^{i_k}.
\end{aligned}
$$

This shows that (3.17) holds for n and integers $r \geqslant 0$.

By (3.3), we get

$$
\begin{aligned}
V(a^{-1};0,0) &= \sum_{i=0}^{n}(-1)^{n+i} a_i^{-n} V_i(a^{-1}) \\
&= \prod_{0\leq i<j\leq n} (a_j^{-1} - a_i^{-1}) \\
&= \prod_{0\leq i\leq n} a_i^{-n} \prod_{0\leq i<j\leq n} (a_i - a_j) \\
&= (-1)^{n(n+1)/2} \prod_{0\leq i\leq n} a_i^{-n} \cdot V(a;0,0),
\end{aligned}
\tag{3.19}
$$

where $a^{-1} = (a_0^{-1}, a_1^{-1}, \cdot, a_n^{-1})$.

If $r < -n$, then $-(n+r) > 0$. From (3.2), (3.19) and proving result above, we find

$$
\begin{aligned}
Va;r,0) &= \prod_{0\leqslant i\leqslant n} a_i^{n-1} \begin{vmatrix} a_0^{-(n-1)} & a_0^{-(n-1)} & \cdots & 1 & a_0^{r+1} \\ a_1^{-(n-1)} & a_1^{-(n-2)} & \cdots & 1 & a_1^{r+1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n-1}^{-(n-1)} & a_{n-1}^{-(n-2)} & \cdots & 1 & a_{n-1}^{r+1} \end{vmatrix} \\
&= (-1)^{n(n-1)/2} \prod_{0\leqslant i\leqslant n} a_i^{n-1} \begin{vmatrix} 1 & a_0^{-1} & \cdots & a_0^{-(n-1)} & a_0^{-[n-(n+r+1)]} \\ 1 & a_1^{-1} & \cdots & a_1^{-(n-1)} & a_1^{-[n-(n+r+1)]} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & a_{n-1}^{-1} & \cdots & a_{n-1}^{-(n-1)} & a_{n-1}^{-[n-(n+r+1)]} \end{vmatrix} \\
&= (-1)^{n(n-1)/2} \prod_{0\leqslant i\leqslant n} a_i^{n-1} \cdot V(a^{-1};0,0) \cdot \sum_{\substack{i_0+i_1+\cdots+i_n=-(n+r+1),\\ i_0,i_1,\cdots,i_n\geq 0 \text{ are integers}}} \prod_{k=0}^{n} a_k^{-i_k} \\
&= (-1)^{n} \prod_{0\leqslant i\leqslant n} a_i^{-1} \cdot V(a;0,0) \cdot \sum_{\substack{i_0+i_1+\cdots+i_n=-(n+r+1),\\ i_0,i_1,\cdots,i_n\geq 0 \text{ are integers}}} \prod_{k=0}^{n} a_k^{-i_k} \\
&= (-1)^{n} V(a;0,0) \cdot \sum_{\substack{i_0+i_1+\cdots+i_n=-r,\\ i_0,i_1,\cdots,i_n\geq 1 \text{ are integers}}} \prod_{k=0}^{n} a_k^{-i_k}.
\end{aligned}
$$

The proof of Lemma 3.2 is completed. □

Lemma 3.3. *Let r be a nonnegative integer, then*

$$
V^2(a;0,0)E_n^{[r]}(a,\lambda) = \sum_{k=0}^{n}(-1)^{k+1}\lambda_k V(a,k;r)V_k(a),
\tag{3.20}
$$

where

$$E_n^{[r]}(a,\lambda)=\sum_{\substack{i_0+i_1+\cdots+i_n=r,\\ i_0,i_1,\cdots,i_n\geq 0 \text{ are integers}}}\left(\sum_{k=0}^{n}(1+i_k)\lambda_k\right)\prod_{k=0}^{n}a_k^{i_k}. \tag{3.21}$$

Proof. Setting a function

$$E_n^{[r]}(a,\lambda)=\sum_{k=0}^{n}\lambda_k B_k(a). \tag{3.22}$$

Let $\lambda_k=1,\lambda_i=0$ $(0\leqslant i\leqslant n, i\neq k)$ in (3.22), from Lemma 3.2 and (3.3), we have

$$\begin{aligned}
B_k&=\sum_{\substack{i_0+i_1+\cdots+i_n=r,\\ i_0,i_1,\cdots,i_n\geq 0 \text{ are integers}}}(1+i_k)\prod_{k=0}^{n}a_k^{i_k}\\
&=\sum_{j=0}^{r}(1+r-j)a_k^{r-j}\sum_{\substack{i_0+i_1+\cdots+i_n=j, i_k=0,\\ i_0,i_1,\cdots,i_n\geqslant 0 \text{ are integers}}}\prod_{v=0}^{n}a_v^{i_v}\\
&=\sum_{j=0}^{r}\left(\sum_{i=0}^{j}a_k^{r-i}\right)\sum_{\substack{i_0+i_1+\cdots+i_n=j, i_k=0,\\ i_0,i_1,\cdots,i_n\geqslant 0 \text{ are integers}}}\prod_{v=0}^{n}a_v^{i_v}\\
&=\sum_{j=0}^{r}a_k^{r-j}\left(\sum_{i=0}^{j}a_k^{j-i}\sum_{\substack{i_0+i_1+\cdots+i_n=j, i_k=0,\\ i_0,i_1,\cdots,i_n\geqslant 0 \text{ are integers}}}\prod_{v=0}^{n}a_v^{i_v}\right)\\
&=\sum_{j=0}^{r}a_k^{r-j}\sum_{\substack{i_0+i_1+\cdots+i_n=j,\\ i_0,i_1,\cdots,i_n\geqslant 0 \text{ are integers}}}\prod_{v=0}^{n}a_v^{i_v}\\
&=\sum_{j=0}^{r}a_k^{r-j}V(a;j,0)/V(a;0,0)=\sum_{j=0}^{n+r}a_k^{j}V(a;r-j,0)/V(a;0,0)\\
&=\sum_{j=0}^{n+r}a_k^{j}\sum_{i=0}^{n}(-1)^{n+i}a_i^{n+r-j}V_i(a)/V(a;0,0)=\sum_{i=0}^{n}(-1)^{n+i}\left(\sum_{j=0}^{n+r}a_k^{j}a_i^{n+r-j}\right)V_i(a)/V(a;0,0)\\
&=\sum_{i=0,i\neq k}^{n}(-1)^{n+i}\frac{a_k^{n+r+1}-a_i^{n+r+1}}{a_k-a_i}\frac{V_i(a)}{V(a;0,0)}+(-1)^{n+k}(n+r+1)a_k^{n+r}\frac{V_k(a)}{V(a;0,0)}.
\end{aligned}$$

That is

$$\begin{aligned}
&V^2(a;0,0)B_k=\sum_{i=0}^{k-1}(-1)^{n+k+i}a_i^{n+r+1}\cdot V_i(a)\prod_{j=0,j\neq i,k}^{n}(a_j-a_i)\\
&+a_i^{n+r+1}\cdot\left[\sum_{i=0}^{k-1}(-1)^{n+k+i+1}V_i(a)\prod_{j=0,j\neq i,k}^{n}(a_j-a_i)+\sum_{i=k+1}^{n}(-1)^{n+k+i}V_i(a)\prod_{j=0,j\neq i,k}^{n}(a_j-a_i)\right]\\
&+\sum_{i=k+1}^{n}(-1)^{n+k+i+1}a_i^{n+r+1}\cdot V_i(a)\prod_{j=0,j\neq i,k}^{n}(a_j-a_i)+(-1)^{n+k}(n+r+1)a_k^{n+r}\cdot V(a;0,0)\\
&=(-1)^{k+1}V(a;k,0)V_k(a),
\end{aligned}$$

from (3.22), we know that (3.20) is true. This is proved. □

The Proof of Theorem 2.1. If $r \in \mathbb{N}$, and taking $\varphi(t) = \prod_{k=1}^{n+1}(k+r)^{-1}t^{n+r+1}$, then $\varphi^{(n+1)}(t) = t^r$. From Lemma 3.3 and Lemma 3.1, we obtain

$$E_n^{[r]}(a,\lambda) = \prod_{k=1}^{n+1}(k+r)\int_E A(\lambda,x)A^r(a,x)dx, \tag{3.23}$$

and

$$\sum_{k=0}^{n}\lambda_k = (n+1)!\int_E A(\lambda,x)dx. \tag{3.24}$$

Let $a^{1/r} = (a_0^{1/r}, a_1^{1/r}, \cdots, a_n^{1/r})$, $A(a^{1/r},x) = \sum_{k=0}^{n} a_i^{1/r}x_i$, $A(\lambda,x) = \sum_{i=0}^{n}\lambda_i x_i$, we have

$$\begin{aligned} H_n^{[r]}(a) &= \frac{1}{\binom{n+r+1}{r}\sum_{k=0}^n \lambda_k} \sum_{\substack{i_0+i_1+\cdots+i_n=r,\\ i_0,i_1,\cdots,i_n\geq 0 \text{ are integers}}} \left(\sum_{k=0}^{n}(1+i_k)\lambda_k\right)\prod_{k=0}^{n} a_k^{i_k/r} \\ &= \frac{E_n^{[r]}(a^{1/r},\lambda)}{\binom{n+r+1}{r}\sum_{k=0}^n \lambda_k} \\ &= \frac{E_n^{[r]}(a^{1/r},\lambda)}{\prod_{k=1}^{n+1}(k+r)}\cdot\frac{(n+1)!}{\sum_{k=0}^n \lambda_k} \\ &= \frac{\int_E (\sum_{i=0}^n \lambda_i x_i)(\sum_{i=0}^n a_i^{1/r}x_i)^r dx}{\int_E (\sum_{i=0}^n \lambda_i x_i)dx}. \end{aligned}$$

The proof of Theorem 2.1 is completed. □

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(Zh.-G. Xiao) Hunan Institute of Science and Technology, Yueyang, Hunan 414006, China
E-mail address: xiaozg@163.com

(Zh.-H. Zhang) Zixing Educational Research Section, Chenzhou, Hunan 423400, China
E-mail address: zxzh1234@163.com

(V. Lokesha) Department of Mathematics, Acharya institute of Technology, Soldevahnalli, Hesaragatta Road, Karanataka Bangalore-90 INDIA
E-mail address: lokiv@yahoo.com