INEQUALITIES FOR STIELTJES INTEGRALS WITH CONVEX INTEGRATORS AND APPLICATIONS

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Abstract. Inequalities for a Grüss type functional in terms of Stieltjes integrals with convex integrators are given. Applications to the Čebyšev functional are also provided.

1. Introduction

In [3], the authors have considered the following functional:

$$D(f;u) := \int_a^b f(x)\,du(x) - [u(b) - u(a)] \cdot \frac{1}{b-a}\int_a^b f(t)\,dt, \tag{1.1}$$

provided that the Stieltjes integral $\int_a^b f(x)\,du(x)$ and the Riemann integral $\int_a^b f(t)\,dt$ exist.

In [3], the following result in estimating the above functional has been obtained:

Theorem 1. *Let $f,u : [a,b] \to \mathbb{R}$ be such that u is Lipschitzian on $[a,b]$, i.e.,*

$$|u(x) - u(y)| \le L|x-y| \quad \text{for any} \quad x,y \in [a,b] \quad (L>0) \tag{1.2}$$

and f is Riemann integrable on $[a,b]$.

If $m,M \in \mathbb{R}$ are such that

$$m \le f(x) \le M \quad \text{for any} \quad x \in [a,b], \tag{1.3}$$

then we have the inequality

$$|D(f;u)| \le \frac{1}{2}L(M-m)(b-a). \tag{1.4}$$

The constant $\frac{1}{2}$ is sharp in the sense that it cannot be replaced by a smaller quantity.

In [2], the following result complementing the above has been obtained:

Theorem 2. *Let $f,u : [a,b] \to \mathbb{R}$ be such that u is of bounded variation on $[a,b]$ and f is Lipschitzian with the constant $K>0$. Then we have*

$$|D(f;u)| \le \frac{1}{2}K(b-a)\bigvee_a^b(u). \tag{1.5}$$

The constant $\frac{1}{2}$ is sharp in the above sense.

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For a function $u:[a,b]\to\mathbb{R}$, define the associated functions Φ,Γ and Δ by:

$$\Phi(t):=\frac{(t-a)u(b)+(b-t)u(a)}{b-a}-u(t),\quad t\in[a,b]; \tag{1.6}$$
$$\Gamma(t):=(t-a)[u(b)-u(t)]-(b-t)[u(t)-u(a)],\quad t\in[a,b]$$

and

$$\Delta(t):=\frac{u(b)-u(t)}{b-t}-\frac{u(t)-u(a)}{t-a},\quad t\in(a,b).$$

In [1], the following subsequent bounds for the functional $D(f;u)$ have been pointed out:

Theorem 3. *Let $f,u:[a,b]\to\mathbb{R}$.*

(i) *If f is of bounded variation and u is continuous on $[a,b]$, then*

$$|D(f;u)|\le\begin{cases}\sup_{t\in[a,b]}|\Phi(t)|\bigvee_a^b(f),\\[2ex] \frac{1}{b-a}\sup_{t\in[a,b]}|\Gamma(t)|\bigvee_a^b(f),\\[2ex] \frac{1}{b-a}\sup_{t\in(a,b)}[(t-a)(b-t)|\Delta(t)|]\bigvee_a^b(f).\end{cases} \tag{1.7}$$

(ii) *If f is $L-$Lipschitzian and u is Riemann integrable on $[a,b]$, then*

$$|D(f;u)|\le\begin{cases}L\int_a^b|\Phi(t)|\,dt,\\[2ex] \frac{L}{b-a}\int_a^b|\Gamma(t)|\,dt,\\[2ex] \frac{L}{b-a}\int_a^b(t-a)(b-t)|\Delta(t)|\,dt.\end{cases} \tag{1.8}$$

(iii) *If f is monotonic nondecreasing on $[a,b]$ and u is continuous on $[a,b]$, then*

$$|D(f;u)|\le\begin{cases}\int_a^b|\Phi(t)|\,df(t),\\[2ex] \frac{1}{b-a}\int_a^b|\Gamma(t)|\,df(t),\\[2ex] \frac{1}{b-a}\int_a^b(t-a)(b-t)|\Delta(t)|\,df(t).\end{cases} \tag{1.9}$$

The case of monotonic integrators is incorporated in the following two theorems [1]:

Theorem 4. *Let $f,u:[a,b]\to\mathbb{R}$ be such that f is $L-$Lipschitzian on $[a,b]$ and u is monotonic nondecreasing on $[a,b]$, then*

$$|D(f;u)|\le\frac{1}{2}L(b-a)[u(b)-u(a)-K(u)] \tag{1.10}$$
$$\le\frac{1}{2}L(b-a)[u(b)-u(a)],$$

where

$$K(u):=\frac{4}{(b-a)^2}\int_a^b u(x)\left(x-\frac{a+b}{2}\right)dx\ge 0. \tag{1.11}$$

The constant $\frac{1}{2}$ in both inequalities is sharp.

Theorem 5. *Let $f,u:[a,b]\to\mathbb{R}$ be such that u is monotonic nondecreasing on $[a,b]$, f is of bounded variation on $[a,b]$ and the Stieltjes integral $\int_a^b f(x)\,du(x)$ exists. Then*

$$|D(f;u)|\le[u(b)-u(a)-Q(u)]\bigvee_a^b(f)\le[u(b)-u(a)]\bigvee_a^b(f), \tag{1.12}$$

where

$$Q(u):=\frac{1}{b-a}\int_a^b \operatorname{sgn}\left(x-\frac{a+b}{2}\right)u(x)\,dx\ge 0. \tag{1.13}$$

The first inequality in (1.12) is sharp.

The main aim of this paper is to establish new sharp inequalities for the functional $D(\cdot;\cdot)$ in the assumption that the integrator u in the Stieltjes integral $\int_a^b f(x)\,du(x)$ is convex on $[a,b]$. Applications for the Čebyšev functional of two Lebesgue integrable function are also given.

2. Inequalities for Convex Integrators

The following result may be stated:

Theorem 6. *Let $u:[a,b]\to\mathbb{R}$ be a convex function on $[a,b]$ and $f:[a,b]\to\mathbb{R}$ a monotonic nondecreasing function on $[a,b]$. Then*

$$\begin{aligned}0&\le D(f;u)\\&\le 2\cdot\frac{u'_-(b)-u'_+(a)}{b-a}\int_a^b\left(t-\frac{a+b}{2}\right)f(t)\,dt\\&\le\begin{cases}\frac{1}{2}\left[u'_-(b)-u'_+(a)\right]\max\{|f(a)|,|f(b)|\}(b-a);\\[2mm] \frac{1}{(q+1)^{\frac{1}{q}}}\left[u'_-(b)-u'_+(a)\right]\|f\|_p(b-a)^{\frac{1}{q}}\\ \qquad\qquad \text{if } p>1,\ \frac{1}{p}+\frac{1}{q}=1;\\[2mm] \left[u'_-(b)-u'_+(a)\right]\|f\|_1.\end{cases}\end{aligned} \tag{2.1}$$

Proof. Integrating by parts in the Stieltjes integral, we have

$$\begin{aligned}\int_a^b\Phi(t)\,df(t)&=\left[\frac{(t-a)u(b)+(b-t)u(a)}{b-a}-u(t)\right]f(t)\Big|_a^b\\&\quad-\int_a^b f(t)\,d\left[\frac{(t-a)u(b)+(b-t)u(a)}{b-a}-u(t)\right]\\&=[u(b)-u(b)]f(b)-[u(a)-u(a)]f(a)\\&\quad-\int_a^b f(t)\left[\frac{u(b)-u(a)}{b-a}dt-du(t)\right]\\&=\int_a^b f(t)\,du(t)-\frac{u(b)-u(a)}{b-a}\int_a^b f(t)\,dt=D(f;u),\end{aligned} \tag{2.2}$$

for any u a continuous function on $[a,b]$ and f of bounded variation on $[a,b]$.

This identity has been established in [1]. In equation (56) in [1], there is a typographical error in the first equation. The definition of Φ is provided in (1.6).

The fact that $D(f;u)\geq 0$ for u convex and f monotonic nondecreasing on $[a,b]$ has been proven earlier in [1]. For the sake of completeness we give here a different and simpler proof as well.

Since u is convex, then

$$\frac{t-a}{b-a}\cdot u(b)+\frac{b-t}{b-a}\cdot u(a)\geq u\left[\frac{(t-a)b+(b-t)a}{b-a}\right] = u(t),$$

for any $t\in[a,b]$. Thus, $\Phi(t)\geq 0$ for $t\in[a,b]$ and since f is monotonic nondecreasing, then $\int_a^b \Phi(t)\,df(t)\geq 0$.

Now, for any convex function $\Phi:[a,b]\to\mathbb{R}$ we have

$$\Phi(x)-\Phi(y)\geq \Phi'_{\pm}(y)(x-y)\quad\text{for any}\quad x,y\in(a,b) \tag{2.3}$$

where $\Phi'_{\pm}$ are the lateral derivatives of the convex function Φ. Then, on using (2.3), we have

$$u'(t)-u(b)\geq u'_{-}(b)(t-b).$$

If we multiply this inequality by $t-a\geq 0$, we get

$$(t-a)u(t)-(t-a)u(b)\geq u'_{-}(b)(t-b)(t-a). \tag{2.4}$$

Similarly, we have

$$(b-t)u(t)-(b-t)u(a)\geq u'_{+}(a)(t-a)(b-t). \tag{2.5}$$

Adding (2.4) with (2.5) and dividing by $b-a$, we deduce:

$$u(t)-\frac{(t-a)u(b)+(b-t)u(a)}{b-a}\geq \frac{(b-t)(t-a)}{b-a}\left[u'_{+}(a)-u'_{-}(b)\right]$$

giving the inequality:

$$0\leq \frac{(t-a)u(b)+(b-t)u(a)}{b-a}-u(t)\leq \frac{(b-t)(t-a)}{b-a}\left[u'_{-}(b)-u'_{+}(a)\right]. \tag{2.6}$$

Integrating this inequality, we get

$$\int_a^b \Phi(t)\,df(t)\leq \frac{\left[u'_{-}(b)-u'_{+}(a)\right]}{b-a}\int_a^b (b-t)(t-a)\,df(t).$$

On the other hand

$$\int_a^b (b-t)(t-a)\,df(t) = f(t)(b-t)(t-a)\Big|_a^b-\int_a^b f(t)\left[-2t+(a+b)\right]dt = 2\int_a^b f(t)\left(t-\frac{a+b}{2}\right)dt,$$

giving the second inequality in (2.1).

Utilising Hölder's inequality, we have

$$\int_a^b \left(t-\frac{a+b}{2}\right) f(t)\,dt \le \begin{cases} \sup\limits_{t\in[a,b]} |f(t)| \int_a^b \left|t-\frac{a+b}{2}\right| dt; \\ \\ \left(\int_a^b |f(t)|^p\,dt\right)^{\frac{1}{p}} \left(\int_a^b \left|t-\frac{a+b}{2}\right|^q dt\right)^{\frac{1}{q}} \\ \qquad\qquad\qquad \text{if } \ p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \\ \sup\limits_{t\in[a,b]} \left|t-\frac{a+b}{2}\right| \int_a^b |f(t)|\,dt, \end{cases}$$

$$= \begin{cases} \frac{1}{4}\max\{|f(a)|,|f(b)|\}(b-a)^2; \\ \\ \frac{1}{2}\cdot\frac{1}{(q+1)^{\frac{1}{q}}}\|f\|_p (b-a)^{1+\frac{1}{q}} \\ \qquad\qquad \text{if } \ p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \\ \frac{1}{2}\|f\|_1 (b-a), \end{cases}$$

and the last part of (2.1) is proved.

Now, for the best possible constant.

Assume that (2.1) holds with a constant C instead of 2, i.e.,

$$D(f;u) \le C\cdot\frac{u'_-(b)-u'_+(a)}{b-a}\int_a^b \left(t-\frac{a+b}{2}\right) f(t)\,dt, \tag{2.7}$$

where u is convex on $[a,b]$ and f is monotonic nondecreasing on $[a,b]$.

Consider $u(t) := \left|t-\frac{a+b}{2}\right|$ and $f(t)=\operatorname{sgn}\left(t-\frac{a+b}{2}\right)$. Then u is convex on $[a,b]$ and f is monotonic nondecreasing on $[a,b]$. We have

$$\begin{aligned} D(f;u) &= \int_a^{\frac{a+b}{2}} (-1)\,d\left(\frac{a+b}{2}-t\right) + \int_{\frac{a+b}{2}}^b (+1)\,d\left(t-\frac{a+b}{2}\right) \\ &= \int_a^b dt = (b-a), \end{aligned}$$

$$u'_-(b)-u'_+(a)=2$$

and

$$\begin{aligned} \int_a^b \left(t-\frac{a+b}{2}\right) f(t)\,dt &= \int_a^b \left(t-\frac{a+b}{2}\right)\operatorname{sgn}\left(t-\frac{a+b}{2}\right) dt \\ &= \int_a^b \left|t-\frac{a+b}{2}\right| dt = \frac{(b-a)^2}{4}. \end{aligned}$$

Therefore, from (2.7) we get

$$b-a \le \frac{C(b-a)}{2},$$

giving that $C\ge 2$. The fact that $\frac{1}{2}$ is best possible goes likewise and we omit the details. ∎

The following result may be stated as well:

Theorem 7. *Let $u:[a,b]\to\mathbb{R}$ be a continuous convex function on $[a,b]$ and $f:[a,b]\to\mathbb{R}$ a function of bounded variation on $[a,b]$. Then*

$$|D(f;u)|\leq\frac{1}{4}\left[u'_{-}(b)-u'_{+}(a)\right](b-a)\bigvee_a^b(f), \tag{2.8}$$

where $\bigvee_a^b(f)$ denotes the total variation of f on $[a,b]$.

The constant $\frac{1}{4}$ is best possible in (2.8).

Proof. It is well known that if $p:[a,b]\to\mathbb{R}$ is continuous on $[a,b]$ and $v:[a,b]\to\mathbb{R}$ is of bounded variation on $[a,b]$, then the Stieltjes integral $\int_a^b p(t)\,dv(t)$ exists and

$$\left|\int_a^b p(t)\,dv(t)\right|\leq\sup_{t\in[a,b]}|p(t)|\bigvee_a^b(f). \tag{2.9}$$

Utilising the inequality (2.6) we have

$$\begin{aligned}&\sup_{t\in[a,b]}\left|\frac{(t-a)u(b)+(b-t)u(a)}{b-a}-u(t)\right|\\&\leq\frac{u'_{-}(b)-u'_{+}(a)}{b-a}\sup_{t\in[a,b]}[(b-t)(t-a)]\\&=\frac{1}{4}(b-a)\left[u'_{-}(b)-u'_{+}(a)\right].\end{aligned}$$

Now, utilising the identity (2.2) and the property (2.9), we have

$$\begin{aligned}|D(f;u)|&\leq\sup_{t\in[a,b]}|\Phi(t)|\bigvee_a^b(f)\\&\leq\frac{1}{4}(b-a)\left[u'_{-}(b)-u'_{+}(a)\right]\end{aligned}$$

and the inequality (2.8) is proved.

Now, for the best constant.

Assume that there exists $D>0$ such that

$$|D(f;u)|\leq D\left[u'_{-}(b)-u'_{+}(a)\right](b-a)\bigvee_a^b(f) \tag{2.10}$$

provided that u is continuous convex and f is of bounded variation on $[a,b]$.

If we choose $u(t)=\left|t-\frac{a+b}{2}\right|$ and $f(t)=\operatorname{sgn}\left(t-\frac{a+b}{2}\right)$, then (see the proof of Theorem 6)

$$D(f;u)=b-a,\quad u'_{-}(b)-u'_{+}(a)=2\quad\text{and}\quad\bigvee_a^b(f)=2$$

giving in (2.10) that $b-a\leq 4D(b-a)$ which implies $D\geq\frac{1}{4}$. ∎

The following result may be stated.

Theorem 8. *Let $u:[a,b]\to\mathbb{R}$ be a convex function on $[a,b]$ and $f:[a,b]\to\mathbb{R}$ a Lipschitzian function with the constant $L>0$, i.e.,*

$$|f(t)-f(s)|\leq L|t-s|\quad\textit{for each}\quad t,s\in[a,b]. \tag{2.11}$$

Then

$$|D(f;u)| \leq \frac{1}{6} L (b-a)^2 \left[u'_-(b) - u'_+(a)\right]. \tag{2.12}$$

Proof. It is well known that if $p : [a,b] \to \mathbb{R}$ is Riemann integrable on $[a,b]$ and $v : [a,b] \to \mathbb{R}$ is Lipschitzian with the constant $L > 0$, then the Stieltjes integral $\int_a^b p(t)\,du(t)$ exists and

$$\left| \int_a^b p(t)\,dv(t) \right| \leq L \int_a^b |p(t)|\,dt. \tag{2.13}$$

Utilising the identity (2.6) and the property (2.13), we have

$$\begin{aligned}
|D(f;u)| &\leq L \int_a^b \left| \frac{(b-t)(t-a)\left[u'_-(b) - u'_+(a)\right]}{b-a} \right| dt \\
&= \frac{L}{b-a} \left[u'_-(b) - u'_+(a)\right] \int_a^b (b-t)(t-a)\,dt \\
&= \frac{1}{6} L (b-a)^2 \left[u'_-(b) - u'_+(a)\right],
\end{aligned}$$

and the theorem is proved. ∎

Remark 1. *It is an open problem if the constant* $\frac{1}{6}$ *above is sharp.*

3. Applications for the Čebyšev Functional

For the Lebesgue integrable functions $f, g : [a,b] \to \mathbb{R}$ with fg an integrable function, consider the *Čebyšev functional* C, defined by

$$C(f,g) = \frac{1}{b-a} \int_a^b f(x)\,g(x)\,dx - \frac{1}{b-a} \int_a^b f(x)\,dx \cdot \frac{1}{b-a} \int_a^b g(x)\,dx.$$

The following result may be stated.

Proposition 1. *If f, g are monotonic nondecreasing functions, then*

$$\begin{aligned}
0 &\leq C(f,g) \\
&\leq 2 \cdot \frac{g(b) - g(a)}{b-a} \cdot \frac{1}{b-a} \int_a^b \left(t - \frac{a+b}{2}\right) f(t)\,dt \\
&\leq \begin{cases}
\frac{1}{2}\left[g(b) - g(a)\right] \max\left\{|f(a)|, |f(b)|\right\}; \\[2mm]
\frac{1}{(q+1)^{\frac{1}{q}}} \left[g(b) - g(a)\right] \|f\|_p (b-a)^{\frac{1}{q}-1} \\
\qquad\qquad \text{if } p > 1,\ \frac{1}{p} + \frac{1}{q} = 1; \\[2mm]
\frac{g(b)-g(a)}{b-a} \|f\|_1 .
\end{cases}
\end{aligned} \tag{3.1}$$

The constants 2 *and* $\frac{1}{2}$ *are best possible.*

The proof is obvious by Theorem 6 on choosing $u : [a,b] \to \mathbb{R}$, $u(t) := \int_a^t g(s)\,ds$. The sharpness of the constant follows as in the proof of Theorem 6 for $f, g : [a,b] = 1$, $f(t) = g(t) = \operatorname{sgn}\left(t - \frac{a+b}{2}\right)$.

The following result may be stated as well:

Proposition 2. *If g is monotonic nondecreasing on $[a,b]$ and f is of bounded variation on $[a,b]$, then*

$$|C(f,g)| \leq \frac{1}{4}[g(b)-g(a)]\bigvee_a^b(f). \tag{3.2}$$

The constant $\frac{1}{4}$ is best possible in (3.2).

The proof follows by Theorem 7 and the details are omitted.

Finally, on utilising Theorem 8, we can state

Proposition 3. *If g is monotonic nondecreasing and f is $L-$Lipschitzian on $[a,b]$, then*

$$|C(f,g)| \leq \frac{1}{6}L(b-a)[g(b)-g(a)].$$

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