ABOUT SURÁNYI'S INEQUALITY

MIHÁLY BENCZE

Abstract. In Miklós Schweitzer Mathematical Competition (Hungary) professor János Surányi proposed the following problem, which is interesting and present an aspect of a theorem. In this paper we present a new demonstration, some interesting applications and a generalization.

Theorem 1. (János Surányi). If $x_k > 0\,(k = 1, 2, ..., n)$ then holds the following inequality:

$$(n-1)\sum_{k=1}^{n} x_k^n + n\prod_{k=1}^{n} x_k \geq \left(\sum_{k=1}^{n} x_k\right)\left(\sum_{k=1}^{n} x_k^{n-1}\right)$$

Proof. Using the mathematical induction for $n = 2$ we obtain $x_1^2+x_2^2+2x_1x_2 \geq (x_1+x_2)^2$ true.

We suppose that is true for n and we prove for $n+1$ too.

Because the inequality is symmetric and homogeneous we can suppose that $x_1 \geq x_2 \geq ... \geq x_{n+1}$ and $x_1+x_2+...+x_n = 1$, so we must prove the following inequality:

$$n\sum_{k=1}^{n+1} x_k^{n+1} + (n+1)\prod_{k=1}^{n+1} x_k \geq \left(\sum_{k=1}^{n+1} x_k\right)\left(\sum_{k=1}^{n+1} x_k^n\right)$$

which can be written in the form

$$n\sum_{k=1}^{n} x_k^{n+1} + nx_{n+1}^{n+1} + nx_{n+1}\prod_{k=1}^{n} x_k + x_{n+1}\prod_{k=1}^{n} x_k - (1+x_{n+1})\left(\sum_{k=1}^{n} x_k^n + x_{n+1}^n\right) \geq 0$$

From the inductive condition holds

$$nx_{n+1}\prod_{k=1}^{n} x_k \geq x_{n+1}\sum_{k=1}^{n} x_k^{n-1} - (n-1)\,x_{n+1}\sum_{k=1}^{n} x_k^n$$

Remain to prove that:

$$\left(n\sum_{k=1}^{n} x_k^{n+1} - \sum_{k=1}^{n} x_k^n\right) - x_{n+1}\left(n\sum_{k=1}^{n} x_k^n - \sum_{k=1}^{n} x_k^{n-1}\right) + x_{n+1}\left(\prod_{k=1}^{n} x_k + (n-1)\,x_{n+1}^n - x_{n+1}^{n-1}\right) \geq 0,$$

but this inequality can be decomposed in two inequalities in following way:

First, from the Csebishef's inequality holds:

$$n\sum_{k=1}^{n} x_k^n - \sum_{k=1}^{n} x_k^{n-1} \geq 0.$$

Second, because $nx_k^{n+1}+\frac{1}{n}x_k^{n-1}\geq 2x_k^n\ (k=1,2,...,n)$, then after addition holds:

$$\begin{aligned}&\prod_{k=1}^{n}x_k+(n-1)\,x_{n+1}^n-x_{n+1}^{n-1}\\&=\prod_{k=1}^{n}(x_k-x_{n+1}+x_{n+1})+(n-1)\,x_{n+1}^n-x_{n+1}^{n-1}\\&\geq x_{n+1}^n+x_{n+1}^{n-1}\sum_{k=1}^{n}(x_k-x_{n+1})+(n-1)\,x_{n+1}^n-x_{n+1}^{n-1}=0\end{aligned}$$

or

$$n\sum_{k=1}^{n}x_k^{n+1}-\sum_{k=1}^{n}x_k^n\geq\frac{1}{n}\left(n\sum_{k=1}^{n}x_k^n-\sum_{k=1}^{n}x_k^{n-1}\right)$$

but from $x_{n+1}\leq\frac{1}{n}$ holds the desired inequality.

If in Theorem 1 we take $n=3$, then we obtain:

Application 1. If $x_1,x_2,x_3\geq 0$, then

$$x_1^3+x_2^3+x_3^3+3x_1x_2x_3\geq x_1^2\,(x_2+x_3)+x_2^2(x_3+x_1)+x_3^2\,(x_1+x_2)$$

which is the well known Schur's inequality. Therefore, the inequality of Surányi generalized the Schur‘s inequality.

Application 2. If a,b,c denote the sides of triangle ABC, s the semiperimeter, R the radius of circumcircle, r the radius of incircle, then:
1). $R\geq 2r$ (the inequality of Euler)
2). $s^2\geq r^2+16Rr$
3). $(4R+r)^3\geq s^2\,(16R-5r)$

Proof. In Application 1 we take:
1). $x_1=a,x_2=b,x_3=c$
2). $x_1=s-a,x_2=s-b,x_3=s-c$
3). $x_1=r_a,x_2=r_b,x_3=r_c$
where r_a,r_b,r_c are the radii of exinscribed circles.

If In Theorem 1 we take $n=4$, then we obtain the following:

Application 3. If $x_1,x_2,x_3,x_4\geq 0$, then

$$2\left(\sum_{k=1}^{4}x_k^4+2\prod_{k=1}^{4}x_k\right)\geq\sum_{1\leq i<j\leq 4}x_ix_j\left(x_i^2+x_j^2\right)$$

Remark. Because $x_i^2+x_j^2\geq 2x_ix_j$, then

$$\sum_{k=1}^{4}x_k^4+2\prod_{k=1}^{4}x_k\geq\sum_{1\leq i<j\leq 4}x_i^2x_j^2,$$

but this is the Turkevici inequality. Therefore the inequality of Surányi give a refinement and a generalization for Turkevici‘s inequality.

Application 4. Denote r_a, r_b, r_c, r_d and h_a, h_b, h_c, h_d the radii of exinscribed spheres and the altitudes in tetrahedron $ABCD$, then
1). $3\sum \frac{1}{h_a^4} + \frac{4}{\prod h_a} \geq \frac{1}{r}\sum \frac{1}{h_a^3}$
2). $3\sum \frac{1}{r_a^4} + \frac{4}{\prod r_a} \geq \frac{2}{r}\sum \frac{1}{r_a^3}$
where r is the radius of inscribed sphere.

Proof. In Application 3 we take:
1). $x_1 = \frac{1}{h_a}, x_2 = \frac{1}{h_b}, x_3 = \frac{1}{h_c}, x_4 = \frac{1}{h_d}$ and $\sum \frac{1}{h_a} = \frac{1}{r}$
2). $x_1 = \frac{1}{r_a}, x_2 = \frac{1}{r_b}, x_3 = \frac{1}{r_c}, x_4 = \frac{1}{r_d}$ and $\sum \frac{1}{r_a} = \frac{2}{r}$

The inequality of Turkevici can be generalized in following way:

Theorem 2. If $x_k > 0\,(k = 1, 2, ..., n)$, then

$$\sum_{1\leq i<j\leq n} (x_i - x_j)^2 + n\sqrt[n]{\prod_{k=1}^{n} x_k^2} \geq \sum_{k=1}^{n} x_k^2$$

And in finally, we generalize the inequality of Surányi in following way:

Theorem 3. If $a_k \in I\ (I \subseteq R)\ (k = 1, 2, ..., n)$, $f : I \to R$ and f and f' are convex functions, then:

$$(n-1)\sum_{k=1}^{n} f(a_k) + nf\left(\frac{1}{n}\sum_{k=1}^{n} a_k\right) \geq \sum_{i,j=1}^{n} f\left(\frac{(n-1)a_i + a_j}{n}\right)$$

Proof. We suppose that $a_1 \geq a_2 \geq ... \geq a_n$, so the desired inequality can be discomposed in the following two inequalities:

$$(1)\quad \sum_{k=1}^{n-1} (n-1-k) f(a_k) + \sum_{k=1}^{n-1} f\left(\frac{ka_k + a_{k+1} + ... + a_n}{n}\right) \geq \sum_{1\leq i<j\leq n} f\left(\frac{(n-1)a_i + a_j}{n}\right)$$

and

$$(2)\quad \sum_{k=1}^{n-1} (k-1) f(a_k) + (n-2) f(a_n) + nf\left(\frac{a_1 + a_2 + ... + a_n}{n}\right) \geq \sum_{k=1}^{n-1} f\left(\frac{ka_k + a_{k+1} + ... + a_n}{n}\right) + \sum_{1\leq i<j\leq n} f\left(\frac{(n-1)a_i + a_j}{n}\right)$$

The inequality (1) is the consequence of inequalities

$$(n-1-k) f(a_k) + f\left(\frac{ka_k + a_{k+1} + ... + a_n}{n}\right) \geq \sum_{j=k+1}^{n} f\left(\frac{(n-1)a_k + a_j}{n}\right),$$

where $k \in \{1, 2, ..., n-1\}$ but this holds from Karamata‘s inequality using for

$$\left(a_k, a_k, ..., a_k, \frac{ka_k + a_{k+1} + ... + a_n}{n}\right) \quad \text{and}$$
$$\left(\frac{(n-1)\,a_k + a_{k+1}}{n}, \frac{(n-1)\,a_k + a_{k+2}}{n}, ..., \frac{(n-1)\,a_k + a_n}{n}\right).$$

The inequality of Karamata say: If $f : I \to R$ is convex $x_1 \geq x_2 \geq ... \geq x_n$ and $y_1 \geq y_2 \geq ... \geq y_n$, $x_1 \geq y_1$,

$$\begin{aligned} x_1 + x_2 \geq y_1 + y_2, ..., x_1 + x_2 + ... + x_{n-1} \\ \geq y_1 + y_2 + ... + y_{n-1}, x_1 + x_2 + ... + x_n \\ = y_1 + y_2 + ... + y_n, \end{aligned}$$

then

$$f\,(x_1) + f\,(x_2) + ... + f\,(x_n) \geq f\,(y_1) + f\,(y_2) + ... + f\,(y_n).$$

In our case

$$(x_1, x_2, ..., x_{n-k}) = \left(a_k, a_k, ..., a_k, \frac{ka_k + a_{k+1} + ... + a_n}{n}\right)$$

and

$$(y_1, y_2, ..., y_{n-k}) = \left(\frac{(n-1)\,a_k + a_{k+1}}{n}, \frac{(n-1)\,a_k + a_{k+2}}{n}, ..., \frac{(n-1)\,a_k + a_n}{n}\right).$$

Now we prove the inequality (2).
Denote

$$\begin{aligned} &F\,(a_1, a_2, ..., a_n) \\ &= \sum_{i=1}^{n-1} (i-1)\,f\,(a_i) + (n-2)\,f\,(a_n) + nf\left(\frac{a_1 + a_2 + ... + a_n}{n}\right) \\ &\quad - \sum_{i=1}^{n-1} f\left(\frac{ia_i + a_{i+1} + ... + a_n}{n}\right) - \sum_{1 \leq i < j \leq n} f\left(\frac{(n-1)\,a_i + a_j}{n}\right), \end{aligned}$$

for which we prove that:

$$\begin{aligned} F\,(a_1, a_2, ..., a_n) &\geq F\,(a_2, a_2, a_3, ..., a_n) \\ &\geq \cdots \\ &\geq F\,(a_{n-1}, a_{n-1}, ..., a_{n-1}, a_n) \\ &\geq F\,(a_n, a_n, ..., a_n) = 0. \end{aligned}$$

In $F\,(a_k, a_k, ..., a_k, a_{k+1}, a_{k+2}, ..., a_n)$, contain a_k the following expression

$$\begin{aligned} &\sum_{i=1}^{n} (i-1)\,f\,(a_k) + nf\left(\frac{ka_k + a_{k+1} + ... + a_n}{n}\right) - \sum_{i=1}^{k} f\left(\frac{ka_k + a_{k+1} + ... + a_n}{n}\right) \\ &\quad - \sum_{1 \leq i < j \leq k} f\left(\frac{(n-1)\,a_k + a_k}{n}\right) - \sum_{j=1}^{k} \sum_{i=k+1}^{n} f\left(\frac{(n-1)\,a_i + a_k}{n}\right) \\ &\quad = (n-k)\,f\left(\frac{ka_k + a_{k+1} + ... + a_n}{n}\right) - k \sum_{i=k+1}^{n} f\left(\frac{(n-1)\,a_i + a_k}{n}\right). \end{aligned}$$

Denote $G_k(a) = F(a, a, ..., a, a_{k+1}, a_{k+2}, ..., a_n)$, where $a \in [a_{k+1}, a_k]$, then

$$G'_k(a) = \frac{k(n-k)}{n}\left(f'\left(\frac{ka + a_{k+1} + ... + a_n}{n}\right) - \frac{1}{n-k}\sum_{i=k+1}^{n} f'\left(\frac{(n-1)a_i + a}{n}\right)\right)$$

$$\geq 0,$$

because

$$\frac{ka + a_{k+1} + ... + a_n}{n} \geq \frac{1}{n-k}\sum_{i=k+1}^{n}\frac{(n-1)a_i + a}{n}$$

or

$$(n-k)a \geq \sum_{i=k+1}^{n} a_i,$$

which is true.

Because f is convex, then f' is increasing but f' is convex, so

$$f'\left(\frac{ka + a_{k+1} + ... + a_n}{n}\right) \geq f'\left(\frac{1}{n-k}\sum_{i=k+1}^{n}\frac{(n-1)a_i + a}{n}\right)$$

$$\geq \frac{1}{n-k}\sum_{i=k+1}^{n} f'\left(\frac{(n-1)a_i + a}{n}\right),$$

which follows from Jensen's inequality.

Therefore G is increasing and

$$F(a_k, a_k, ..., a_k, a_{k+1}, a_{k+2}, ..., a_n) \geq F(a_{k+1}, a_{k+1}, ..., a_{k+1}, a_{k+2}, ..., a_n)$$

which prove the affirmation.

Remark. If in Theorem 3 we take $f(a) = e^{na}$ and $e^{a_k} = x_k$ $(k = 1, 2, ..., n)$, then we obtain the inequality of Surányi.

Application 5. If $a_k > 0\,(k = 1, 2, ..., n)$ and $\alpha \geq 2$, then

$$(n-1)\sum_{k=1}^{n} a_k^{\alpha} + n\left(\frac{1}{n}\sum_{k=1}^{n} a_k\right)^{\alpha} \geq \sum_{i,j=1}^{n}\left(\frac{(n-1)a_i + a_j}{n}\right)^{\alpha}.$$

Proof. In Theorem 3 we take $f(a) = a^{\alpha}$.

References.

[1]. Mihály Bencze: Inequalities (manuscript), 1982.

[2]. D.S. Mitrinović, J.E. Pečarić, A.M. Fink: Classical and New Inequalities in Analysis, Kluwer Academic Publishers, 1993.

[3]. Octogon Mathematical Magazine (1993-2005).

Str. Hărmanului 6, 505600 Săcele, Jud. Braşov, Romania