MINKOWSKI, HÖLDER AND CHEBYSHEV TYPE INEQUALITY OF HOMOGENEOUS FUNCTIONS

ZHEN-HANG YANG

ABSTRACT. In this paper, we study the AA, GG, AG and GA convexities of homogeneous functions of two variables, of which simplified decision methods are presented. As applications, new simple proofs of H–type, Ch–type and M–type inequalities for homogeneous means or functions are given.

1. INTRODUCTION

Suppose $x_1, x_2, y_1, y_2 > 0$, it is well-known for arithmetic mean of two positive numbers x and y denoted by $f(x, y) = A(x, y) = \frac{x+y}{2}$, there is Hölder inequality:

$$f(x_1^p x_2^q, y_1^p y_2^q) \leq f^p(x_1, y_1) f^q(x_2, y_2), \tag{1.1}$$

where $pq > 0$ with $p + q = 1$. It is reversed for $pq < 0$ with $p + q = 1$. with equality iff $x_1, : y_1 = x_2, y_2$. There is also Chebyshev inequality:

$$f(x_1, y_1) f(x_2, y_2) \leq f(x_1 x_2, y_1 y_2) \tag{1.2}$$

if (x_1, y_1) and (x_2, y_2) are similarly ordered; it is reversed if (x_1, y_1) and (x_2, y_2) are oppositely ordered.

Let $f(x, y) = A^{\frac{1}{p}}(x^p, y^p)$. Then there is also Minkowski inequality:

$$f((x_1 + x_2, y_1 + y_2) \leq f(x_1, y_1) + f(x, y_2) \text{ if } p > 1. \tag{1.3}$$

If a function of two variable $f(x, y)$ satisfies (1),(2),(3), then call them Holder, Minkowski and Chebyshev type inequality, simply call H-type, Ch-type and M-type inequality.

For one-parameter mean of two positive numbers x and y denoted by

$$J(\alpha; x, y) = \begin{cases} \frac{\alpha(x^{\alpha+1} - y^{\alpha+1})}{(\alpha+1)(x^\alpha - y^\alpha)}, & \alpha \neq 0, -1, x \neq y; \\ \frac{x-y}{\ln x - \ln y}, & \alpha = 0, x \neq y; \\ \frac{xy(\ln x - \ln y)}{x-y}, & \alpha = -1, x \neq y; \\ y & x = y \end{cases} \tag{1.4}$$

A. Horst presented M-type and Ch-type inequality in 1988 (see [1, 2].

Date: October 23, 2005.

2000 *Mathematics Subject Classification.* Primary 26B25, 26D07; Secondary 26E60, 26A48 .

Key words and phrases. Homogeneous functions, inequality, convexity.

For generalized logarithmic mean of two positive numbers a and b denoted by

$$S(\alpha;x,y)=\begin{cases}(\frac{x^\alpha-y^\alpha}{\alpha(x-y)})^{\frac{1}{\alpha-1}} & \alpha\neq 0,1,x\neq y;\\ L(x,y) & \alpha=0,x\neq y;\\ E(x,y) & \alpha=1,x\neq y;\\ y & x=y.\end{cases}\tag{1.5}$$

Hongwei Lou proved that the H-type inequality of $S(\alpha;x,y)$ is valid in 1996 (see [6]). Zhen-Hang Yang proved that its H-type, Ch-type and M-type inequalities are also true by using classical integral inequalities in 2005 (see [9])

The so-called extended mean value of positive numbers x, y is denoted by

$$E(r,s;x,y)=\begin{cases}\left(\frac{s}{r}\frac{x^r-y^r}{x^s-y^s}\right)^{\frac{1}{r-s}} & r\neq s, rs\neq 0\\ L^{\frac{1}{r}}(x^r,y^r) & r\neq 0,s=0\\ L^{\frac{1}{s}}(x^s,y^s) & r=0,s\neq 0\\ E^{\frac{1}{r}}(x^r,y^r) & r=s\neq 0\\ G(x,y) & r=s=0\end{cases}\tag{1.6}$$

where

$$L(x,y)=\frac{x-y}{\ln x-\ln y}, E(x,y)=e^{-1}(\frac{x^x}{y^y})^{\frac{1}{y-x}}, G(x,y)=\sqrt{xy}.$$

The Gimi mean is denoted by

$$G(r,s;x,y)=\begin{cases}\left(\frac{x^p+y^p}{x^q+y^q}\right)^{\frac{1}{p-q}} & p\neq q\\ Z^{\frac{1}{p}}(x^p,y^p) & p=q\neq 0\\ G(x,y) & p=q=0\end{cases},\tag{1.7}$$

where $Z(x,y)=x^{\frac{x}{x+y}}y^{\frac{y}{x+y}}$.

In 1998, L. Losonczi and Zs. Páles showed that M-type inequality (2) hold for Stolarsky mean (extended mean) and Gini mean (see [5]).

Note (1.1) and (1.3) can be changed as

$$\ln f(e^{p\ln x_1+q\ln x_2},e^{p\ln y_1+q\ln y_2}) \leq p\ln f(e^{\ln x_1},e_1^{\ln y})+qf(e^{\ln x_2},e^{\ln y_2}),\tag{1.8}$$

$$f((\frac{x_1+x_2}{2},\frac{y_1+y_2}{2}) \leq \frac{1}{2}[f(x_1,y_1)+f(x,y_2)] \text{ if } p>1,\tag{1.9}$$

which show that $\ln f(e^x,e^{y)}$ and $f(x,y)$ are both convex. Thus getting H-type and M-type inequalities, it is enough to confirm the convexity of $\ln f(e^x,e^{y)}$ and $f(x,y)$ respectively. Depending on which type of mean, arithmetic (A) or geometric (G), we consider respectively on the domain and the codomain of definition, we shall encounter one of the following four classes of functions:

$AA-$ convex functions, namely, the usual convex functions;

$GG-$ convex functions, is called geometric convex functions (see [11]);

$AG-$ convex functions, namely, the log-convex functions;

$AG-$ convex functions.

But the AA-convexity is most basic, so for the sake of simplicity, we read the definition and decision theorem of $AA-$convex functions of two variable as follows (see [4]):

Definition 1. *Let* $f : \mathbb{D} \to \Omega$ *is a function of two variable, where* $\mathbb{D}, \Omega \subseteqq \mathbb{R} \times \mathbb{R}$ *and* $\mathbb{D}$ *is a convex codomain. For arbitrary pairs* $(x_1, y_1), (x_2, y_2) \in \mathbb{D}$ *if*

$$f((\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}) \leq \frac{1}{2}[f(x_1, y_1) + f(x, y_2)] \tag{1.10}$$

is always valid, then $f(x, y)$ *is said convex on* $\mathbb{D}$*. If* $-f(x, y)$ *is convex on* $\mathbb{D}$*, then* $f(x, y)$ *is said concave on* $\mathbb{D}$*.*

The Definition above has an equivalent definition, that is:

Definition 2. *Let* $f : \mathbb{D} \to \Omega$ *is a function of two variable, where* $\mathbb{D}, \Omega \subseteqq \mathbb{R} \times \mathbb{R}$ *and* $\mathbb{D}$ *is a convex codomain. For arbitrary pairs* $(x_1, y_1), (x_2, y_2) \in \mathbb{D}$ *and real number* t*, if* $(x_1 + tx_2, y_1 + ty_2) \in \mathbb{D}$ *and*

$$F(t) = f(x_1 + tx_2, y_1 + ty_2)$$

is convex (concave) in t*, then call* $f(x, y)$ *a convex (concave) function defined on* $\mathbb{D}$*.*

Thus it can be seen that decision for convexity of functions of two variables may be came down to one variable. there are

Criterion 1. *Suppose* $f(x, y)$ *is two-time differentiable on open domain* $\mathbb{D}$*, then* $f(x, y)$ *is convex on* $\mathbb{D}$ *iff*

$$F''(t) = f_{xx}{x_2}^2 + 2f_{xy}x_2y_2 + f_{yy}{y_2}^2 \geq 0 \tag{1.11}$$

for all $(x_2, y_2), (x_1 + tx_2, y_1 + ty_2) \in \mathbb{D}$*.*

If $F''(t) > 0$ *unless* $x_2 = y_2 = 0$*, then (1.10) is strict unless* $(a_1, b_1) = (a, b_2)$*.*

However, comparing with the case of one variable, it is more complex and difficult that confirming the convexity of function of two variable. Then whether there exist some more simple decision methods if the function of two variable satisfies certain special conditions? The answer is sure.

The main aim of this paper is to investigate the convexity of functions of two variable under the conditions of homogeneity, and present a simplified decision method, naturally give necessary or sufficient conditions of H-type, Ch-type and M-type inequalities.

This paper is organized as follows. Section 2 recalls the definition and properties of homogeneous functions. Section 3 presents some simplified decision methods of homogeneous functions of two variable, which is main results. Section 4 gives some applications include H-type, Ch-type and M-type inequalities of some well-known means and other new inequalities.

2. Properties of Homogeneous Functions

In order to prove the main results in this paper, we first recall the definition and properties of homogeneous functions.

Definition 3. *Suppose $f : \mathbb{D} \to \Omega$ is a function of two variable, where $\mathbb{D}, \Omega \subseteqq \mathbb{R} \times \mathbb{R}$. For arbitrary $(x, y) \in \mathbb{D}$ and $t > 0$, if $(tx, ty) \in \mathbb{D}$, and*

$$f(tx, ty) = t^n f(x, y), \tag{2.1}$$

then call $f(x, y)$ an n-order homogeneous function defined on $\mathbb{D}$.

Property 1 Let $f(x, y)$, $g(x, y)$ be an n, m-order homogenous function over $\mathbb{D}$ respectively. Then $f \cdot g, f/g(g \neq 0)$ are $n+m, n-m$-order homogenous function over $\mathbb{D}$ respectively.

If for a certain p with $(x^p, y^p) \in \mathbb{D}, f^p(x, y)$ exists, then $f(x^p, y^p), f^p(x, y)$ are both np-order homogeneous functions over $\mathbb{D}$.

Property 2 Let $f(x, y)$ be an n-order homogeneous function over $\mathbb{D}$ and f_x, f_y both exist. Then f_x, f_y are both $(n-1)$-order homogeneous functions over $\mathbb{D}$.

Property 3 Let $f(x, y)$ be an n-order homogeneous function over $\mathbb{D}$ and f_x, f_y both exist. Then

$$xf_x + yf_y = nf. \tag{2.2}$$

In particular, for $n = 1$ and $f(x, y)$ is one-time differentiable over $\mathbb{D}$, then

$$xf_x + yf_y = f, \tag{2.3}$$

$$xf_{xx} + yf_{xy} = 0, \tag{2.4}$$

$$xf_{xy} + yf_{yy} = 0. \tag{2.5}$$

Property 4 Let $f(x, y)$ be an n-order homogeneous function defined on $\mathbb{D}$ and $[0, x] \times [0, y] \subseteqq \mathbb{D}$. If $f(x, y)$ is integrable with respect to x (or y) on $[0, x]$ (or $[0, y]$), then $\int_0^x f(u, y)du$ (or $\int_0^y f(x, v)dv$) is an $n+1$-order homogeneous function.

3. Decisions of Convexity

3.1. **AA–convexity and M–type inequality.**

Let us consider first AA convexity.

Theorem 1. *Let $f : \mathbb{D} \to \Omega$ is a one-order homogeneous function and is two-time differentiable. Then $f(u, v)$ is strictly convex (concave) iff $uvf_{uv} < (>)0$.*

Proof. Let $F(t) = f(u_1 + tu_2, v_1 + tv_2), u_1 + tu_2 = u, v_1 + tv_2 = v$. By Property 3, there are $f_{uu} = -\frac{v}{u} f_{uv}, f_{vv} = -\frac{u}{v} f_{uv}$ for $uv \neq 0$, then

$$\begin{aligned} F''(t) &= f_{uu}u_2^2 + 2f_{uv}u_2v_2 + f_{vv}v_2^2 = -\frac{v}{u} f_{uv}u_2^2 + 2f_{uv}u_2v_2 + -\frac{u}{v} f_{uv}v_2^2 \\ &= -\frac{f_{uv}}{uv}(v^2u_2^2 - 2uvu_2v_2 + u^2v_2^2) = -\frac{f_{uv}}{uv}(v_1u_2 - u_1v_2)^2 \end{aligned}$$

This shows that $F''(t) > 0$ iff $uvf_{uv} < 0$ unless $v_1u_2 - u_1v_2 \neq 0$.

It follows that this Theorem is true. ∎

By Theorem above we derive immediately

Corollary 1 (M–type Inequality)**.** *Let $f : \mathbb{D} \to \mathbb{R}$ is a one-order homogeneous function and is two-time differentiable. Then for given $pq > 0$ with $p+q = 1$ and arbitrary pairs $(x_1, y_1), (x_2, y_2), (px_1 + qx_2, py_1 + qy_2) \in \mathbb{D}$, the following inequality*

$$f(px_1 + qx_2, py_1 + qy_2) \le pf(x_1, y_1) + qf(x_2, y_2) \tag{3.1}$$

holds iff $xyf_{xy} < 0$. With equality iff $x_1 : y_1 = x_2 : y_2$.

Proof. 1) If $I < 0$. By Theorem 1, the $F(t) = f(u_1 + tu_2, v_1 + tv_2)$ with $u_1 : v_1 \ne u_2 : v_2$ is strictly convex, it follows that

$$F(pt_1 + qt_2) \le pF(t_1) + qF(t_2) \tag{3.2}$$

i.e.

$$f(u_1{+}(pt_1{+}qt_2)u_2, v_1{+}(pt_1{+}qt_2)v_2) \le pf(u_1{+}t_1u_2, v_1{+}t_1v_2){+}qf(u_1{+}t_2u, v_1{+}t_2v_2). \tag{3.3}$$

holds for $pq > 0$ with $p + q = 1$. Put

$$(u_1 + t_1u_2, v_1 + t_1v_2) = (x_1, y_1), (u_1 + t_2u_2, v_1 + t_2v_2) = (x_2, y_2), \tag{3.4}$$

then inequality is changed into (3.1).

With equality iff $t_1 = t_2$ or $u_1 : v_1 = u_2 : v_2$ i.e. $x_1 : y_1 = x_2 : y_2$.

If (3.1) holds and With equality iff $x_1 : y_1 = x_2 : y_2$. For arbitrary two pairs (x_1, y_1) and (x_2, y_2) there must exist unequal (u_1, v_1) and $(u_2, , v_2)$ respectively, such that (3.4) hold, and then (3.3) holds i.e. (3.2) holds, which shows that $F(t) = f(u_1 + tu_2, v_1 + tv_2)$ is strictly convex and lead to $xyf_{xy} < 0$ by Theorem (1).

This completes the proof. ∎

Choosing $x_1 = y_2 = x, x_2 = y_1 = y$ in Corollary 1, we have

Corollary 2. *Let $f : \mathbb{D} \to \mathbb{R}$ is a one-order symetric homogeneous function and is two-time differentiable. Then for given $p, q > 0$ with $p + q = 1$ and arbitrary pairs $(x, y), (y, x), (px + qy, qx + py) \in \mathbb{D}$, we have*

$$f(px + qy, qx + py) \le f(x, y) \tag{3.5}$$

if $xyf_{xy} < 0$. It is reversed if $xyf_{xy} > 0$. With equality iff $x = y$.

In particular, if $(1, 1) \in \mathbb{D}$, then we have

$$\frac{x + y}{2} f(1.1) \le f(x, y) \tag{3.6}$$

holds if $xyf_{xy} < 0$. It is reversed if $xyf_{xy} > 0$. With equality iff $x = y$.

3.2. **GG–Convexity and H–type, Ch–type inequalities.** Next we will give the decision theorem of GG-convexity of homogeneous functions of two variable.

Theorem 2. *Let $f : \mathbb{D}(\subsetneqq \mathbb{R}_+ \times \mathbb{R}_+) \to \mathbb{R}_+$ is an n-order homogeneous and two-time differentiable function. Then $g(u, v) = \ln f(e^u, e^v)$ is strictly convex (concave) iff $I = (\ln f)_{xy} < (>)0$.*

Proof. Put $e^u = x, e^v = y$, then

$$g_u = \frac{f_x(x,y)}{f(x,y)}\frac{\mathrm{d}\,x}{\mathrm{d}\,u} = x\frac{f_x(x,y)}{f(x,y)},$$
$$g_v = \frac{f_y(x,y)}{f(x,y)}\frac{\mathrm{d}\,y}{\mathrm{d}\,v} = y\frac{f_y(x,y)}{f(x,y)},$$

hence

$$g_u + g_v = n.$$

From Property 3 it follows that

$$g_{uu} + g_{vu} = 0 \text{ i.e. } g_{uu} = -g_{vu},$$
$$g_{uv} + g_{vv} = 1 \text{ i.e. } g_{vv} = -g_{uv}.$$

Let $G(t) = g(u,v)$ where $(u,v) = (u_1 + tu_2, v_1 + tv_2)$. Then

$$G''(t) = g_{uu}u_2^2 + 2g_{uv}u_2v_2 + g_{vv}v_2^2 = -g_{uv}(u_2^2 - 2u_2v_2 + v_2^2) = -g_{uv}(u_2 - v_2)^2. \tag{3.7}$$

This shows $G''(t) > 0$ iff $g_{uv} < 0$ unless $u_2 = v_2$, where

$$g_{uv} = \frac{\partial g_u}{\partial v} = \frac{\partial g_u}{\partial y}\frac{\mathrm{d}\,y}{\partial v} = \left(x\frac{f_x(x,y)}{f(x,y)}\right)_y e^v = xy(\ln f(x,y))_{xy} = xyI, \tag{3.8}$$

which implies that $g_{uv} < 0$ iff $I = (\ln f)_{xy} < 0$.

This proof is completed. ∎

Corollary 3 (H–type inequality). *Let $f : \mathbb{D}(\subseteqq \mathbb{R}_+ \times \mathbb{R}_+) \to \mathbb{R}_+$ is an n-order homogeneous and two-time differentiable function. Then for given $p, q > 0$ with $p + q = 1$ and arbitrary pairs $(x_1, y_1), (x_2, y_2), (x_1^p x_2^q, y_1^p y_2^q) \in \mathbb{D}$, inequality*

$$f(x_1^p x_2^q, y_1^p y_2^q) \le f^p(x_1, y_1) f^q(x_2, y_2) \tag{3.9}$$

holds iff $I = (\ln f)_{xy} < 0$. With equality iff $x_1 : y_1 = x_2 : y_2$.

Proof. 1) If $I < 0$. By Theorem 2, the $G(t) = g(u_1 + tu_2, v_1 + tv_2)$ with $u_2 \ne v_2$ is strictly convex, it follows that

$$G(pt_1 + qt_2) \le pG(t_1) + qG(t_2) \tag{3.10}$$

i.e.

$$\ln f(e^{u_1+(pt_1+qt_2)u_2}, e^{v_1+(pt_1+qt_2)v_2}) \le p\ln f(e^{u_1+t_1u_2}, e^{v_1+t_1v_2}) + q\ln f(e^{u_1+t_2u_2}, e^{v_1+t_2v_2}). \tag{3.11}$$

holds for $pq > 0$ with $p + q = 1$. Put

$$(e^{u_1+t_1u_2}, e^{v_1+t_1v_2}) = (x_1, y_1), (e^{u_1+t_2u_2}, e^{v_1+t_2v_2}) = (x_2, y_2), \tag{3.12}$$

then inequality (3.11) is changed as

$$\ln f(x_1^p x_2^q, y_1^p y_2^q) \le p\ln f(x_1, y_1) + q\ln f(x_2, y_2), \tag{3.13}$$

Removing the logarithmic signs of two sides in inequality (3.13) (3.9) is derived.

With equality iff $t_1 = t_2$ or $u_2 = v_2$ i.e. $x_1 : y_1 = x_2 : y_2$.

If (3.9) holds and With equality iff $x_1 : y_1 = x_2 : y_2$. For arbitrary two pairs (x_1, y_1) and (x_2, y_2) there must exist (u_1, v_1) and (u_2, v_2) respectively, such that (3.12) holds, and then (3.11) holds i.e. (3.10) holds, which shows that $G(t) = g(u_1 + tu_2, v_1 + tv_2)$ is strictly convex and results in $I < 0$ by Theorem 2.

This completes the proof. ∎

Similar to Corollary 2, we have

Corollary 4. *Let $f : \mathbb{D}(\subseteqq \mathbb{R}_+ \times \mathbb{R}_+) \to \mathbb{R}_+$ is an n-order homogeneous and two-time differentiable function. Then for given $p, q > 0$ with $p + q = 1$ and arbitrary pairs $(x, y), (y, x), (px + qy, qx + py) \in \mathbb{D}$, there is*

$$f(x^p y^q, x^q y^p) \le f(x, y) \tag{3.14}$$

if $I = (\ln f)_{xy} < 0$. It is reversed if $I = (\ln f)_{xy} > 0$. With equality iff $x = y$.

In particular, if $(1, 1) \in \mathbb{D}$, then there is

$$\sqrt{xy} f(1.1) \le f(x, y) \tag{3.15}$$

if $I = (\ln f)_{xy} < 0$. It is reversed if $I = (\ln f)_{xy} > 0$. With equality iff $x = y$.

Remark 1. *Let $f : \mathbb{D}(\subseteqq \mathbb{R}_+ \times \mathbb{R}_+) \to \mathbb{R}_+$ is a one-order homogeneous and two-time differentiable function. then by (3.6) and (3.15) we have*

$$\sqrt{xy} \le \frac{f(x, y)}{f(1.1)} \le \frac{x + y}{2}. \tag{3.16}$$

if $f_{xy} > 0, I = (\ln f)_{xy} < 0$. It can be concluded that $\frac{f(x,y)}{f(1.1)}$ must be a mean of x and y, and lie in the $\sqrt{xy}$ and $\frac{x+y}{2}$.

Applying the Corollary 3, we can prove the monotonicity of the so-called two-parameter homogeneous functions defined by [10]

$$\mathcal{H}_f(p, q; a, b) = \begin{cases} \left[\dfrac{f(a^p, b^p)}{f(a^q, b^q)}\right]^{\frac{1}{p-q}}, & p \ne q, pq \ne 0; \\ a^{\frac{a^p f_x(a^p, b^p)}{f(a^p, b^p)}} b^{\frac{b^p f_y(a^p, b^p)}{f(a^p, b^p)}}, & p = q \ne 0; \\ \left[\dfrac{f(a^p, b^p)}{f(1, 1)}\right]^{\frac{1}{p}}, & p \ne 0, q = 0; \\ \left[\dfrac{f(a^q, b^q)}{f(1, 1)}\right]^{\frac{1}{q}}, & p = 0, q \ne 0; \\ a^{\frac{f_x(1,1)}{f(1,1)}} b^{\frac{f_y(1,1)}{f(1,1)}}, & p = q = 0. \end{cases}, \tag{3.17}$$

in which f: $\mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ is an n-order homogeneous function for variables x and y, and is continuous and 1st partial derivatives exist, $(a, b) \in \mathbb{R}_+ \times \mathbb{R}_+$ with $a \ne b$, $(p, q) \in \mathbb{R} \times \mathbb{R}$.

Corollary 5. *If $I = (\ln f)_{xy} < 0$, then*

$$\mathcal{H}_f(p_1, q; a, b) < \mathcal{H}_f(p_2, q; a, b) \text{ if } p_1 < p_2. \tag{3.18}$$

It is reversed if $I = (\ln f)_{xy} > 0$.

Proof. Without loss of generality, we assume $pq(p - q) \ne 0$, then 3.18 is equivalent to

$$[f(a^{p_1}, b^{p_1})]^{\frac{p_2 - q}{p_2 - p_1}} [f(a^{p_2}, b^{p_2})]^{\frac{q - p_1}{p_2 - p_1}} > f(a^q, b^q) \text{ if } p_1 < q < p_2, \tag{3.19}$$

and inequality reverses if $q > p_2$ or $q < p_1$.

Using Corollary 3, for $p_1 < q < p_2$ we have

$$\begin{aligned}[f(a^{p_1}, b^{p_1})]^{\frac{p_2-q}{p_2-p_1}}[f(a^{p_2}, b^{p_2})]^{\frac{q-p_1}{p_2-p_1}} &> f(a^{p_1\frac{p_2-q}{p_2-p_1}}a^{p_2\frac{q-p_1}{p_2-p_1}}, b^{p_1\frac{p_2-q}{p_2-p_1}}b^{p_2\frac{q-p_1}{p_2-p_1}}) \\ &= f(a^{p_1\frac{p_2-q}{p_2-p_1}+p_2\frac{q-p_1}{p_2-p_1}}, b^{p_1\frac{p_2-q}{p_2-p_1}+p_2\frac{q-p_1}{p_2-p_1}}) \\ &= f(a^q, b^q);\end{aligned}$$

likewise it is reversed if $q > p_2$ or $q < p_1$.

Thus we end the proof. ∎

Corollary 6 (Ch–type inequality)**.** *Let $f : \mathbb{D}(\subseteqq \mathbb{R}_+ \times \mathbb{R}_+) \to \mathbb{R}_+$ is an n-order homogeneous and two-time differentiable function. If $I = (\ln f(x, y))_{xy} < (>)0$, then for arbitrary pairs $(x_1, y_1), (x_2, y_2)$ and $(1, 1) \in \mathbb{D}$ we have*

$$f(1,1)f(x_1x_2, y_1y_2) < (>)f(x_1, y_1)f(x_2, y_2). \tag{3.20}$$

if (x_1, y_1) and (x_2, y_2) are oppositely ordered. It is reversed if (x_1, y_1) and (x_2, y_2) are similarly ordered

Proof. By Corollary 4, if $I = (\ln f(x, y))_{xy} < (>)0$ for $pq > 0$ with $p+q = 1$ and $x_1 : y_1 \neq x_2 : y_2$ then inequality (3.9) is valid. Likewise we have

$$f(x_1^q x_2^p, y_1^q y_2^p) < (>)f^q(x_1, y_1)f^p(x_2, y_2) \tag{3.21}$$

Let the left and right sides of inequality (3.9) and (3.21) multiply respectively. Then we get

$$f(x_1^p x_2^q, y_1^p y_2^q)f(x_1^q x_2^p, y_1^q y_2^p) < (>)f(x_1, y_1)f(x_2, y_2)$$

choosing $p = \ln\frac{x_1}{y_1}/\ln\frac{x_1y_2}{x_2y_1}, q = -\ln\frac{x_2}{y_2}/\ln\frac{x_1y_2}{x_2y_1}$ in the above inequality, we obtain (3.20) immediately. Here $p \cdot q = -\ln\frac{x_1}{y_1} \cdot \ln\frac{x_2}{y_2}/(\ln\frac{x_1y_2}{x_2y_1})^2 > 0$ which is equivalent to $\ln\frac{x_1}{y_1} \cdot \ln\frac{x_2}{y_2} < 0$ i.e. (x_1, y_1) and (x_2, y_2) are oppositely ordered.

Obviously inequality (3.20) is reversed if (x_1, y_1) and (x_2, y_2) are similarly ordered.

This proof is completed. ∎

Remark 2. *Inequality (3.20) is similar to Chebyshev's, which is just Ch-type inequality of homogeneous functions.*

3.3. **AG–convexity.**

Theorem 3. *Suppose $f : \mathbb{D} \to \mathbb{R}_+$ is a one-order homogeneous and two-time differentiable function. Then $g(x, y) = \ln f(x, y)$ is concave iff $xyf_{xy} > 0$.*

Proof. By simple calculations, we can get

$$\begin{aligned} g_x &= (\ln f)_x = \frac{f_x(x,y)}{f(x,y)}, \\ g_y &= (\ln f)_y = \frac{f_y(x,y)}{f(x,y)}, \\ g_{xx} &= (\ln f)_{xx} = \frac{f(x,y)f_{xx}(x,y) - f_x^2(x,y)}{f^2(x,y)} \\ &= \frac{f(x,y)(-\frac{xy}{x^2})f_{xy}(x,y) - f_x^2(x,y)}{f^2(x,y)}. \end{aligned}$$

And then we have the following equation: $xg_x + yg_y = 1$. From it we get

$$\begin{aligned} xg_{xx} + yg_{yx} &= -g_x, \\ xg_{xy} + yg_{yy} &= -g_y. \end{aligned}$$

Observe that

$$\begin{aligned} g_{xy}^2 - g_{xx}g_{yy} &= g_{xy}^2 - \frac{yg_{xy} + g_x}{x}\frac{xg_{xy} + g_y}{y} \\ &= -\frac{g_{xy}(xg_x + yg_y) + g_xg_y}{xy} \\ &= -\frac{g_{xy} + g_xg_y}{xy} \\ &= -\frac{(\ln f)_{xy} + \frac{f_x(x,y)}{f(x,y)}\frac{f_y(x,y)}{f(x,y)}}{xy} \\ &= -\frac{f_{xy}}{xyf} = -\frac{xyf_{xy}}{(xy)^2 f}, \end{aligned}$$

since $g(x,y) = \ln f(x,y)$ is convex iff $g_{xy}^2 - g_{xx}g_{yy} \le 0$ and $g_{xx} \ge 0$, which is impossible because from $xyf_{xy} > 0$ it follows that $f(x,y)(-\frac{xy}{x^2})f_{xy}(x,y) - f_x^2(x,y) < 0$. Hence $g(x,y) = \ln f(x,y)$ is always concave if $xyf_{xy} > 0$ and vice versa.

This proof ends. ∎

Remark 3. *If $f(x,y)$ is a one-order homogeneous function, from processes of proof of Theorem 3, we see that the convexity of $\ln f(x,y)$ is uncertain.*

Remark 4. *Theorem 3 can be deduced from Theorem 1, because if $xyf_{xy} > 0$ then*

$$f(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}) \ge \frac{1}{2}\left[f(x_1,y_1) + f(x_2,y_2)\right] \ge \sqrt{f(x_1,y_1) + f(x_2,y_2)},$$

i.e.

$$\ln f(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}) \ge \frac{1}{2}\left[\ln f(x_1,y_1) + \ln f(x_2,y_2)\right],$$

which implies $\ln f(x,y)$ is concave on $\mathbb{D}$.

3.4. **GA–Convexity.**

Theorem 4. *Suppose $f : \mathbb{D}(\subsetneqq \mathbb{R}_+ \times \mathbb{R}_+) \to \mathbb{R}$ is a zero-order homogeneous and two-time differentiable function, then $g(u,v) = f(e^u, e^v)$ is convex (concave) iff $f_{xy} < (>) 0$.*

Proof. Put $e^u = x, e^v = y$, by a simply derivative calculation, we have

$$g_u = e^u f_x(e^u, e^v) = xf_x(x,y), g_v = e^v f_y(x,y) = yf_y(x,y).$$

It follows that $g_u + g_v = 0$, and then

$$\begin{aligned} g_{uu} + g_{vu} &= 0 \text{ i.e. } g_{uu} = -g_{vu}, \\ g_{uv} + g_{vv} &= 1 \text{ i.e. } g_{vv} = -g_{uv}. \end{aligned}$$

Let $G(t) = g(u_1 + tu_2, v_1 + tv_2)$. Then

(3.22)

$$G''(t) = g_{uu}u_2^2 + 2g_{uv}u_2v_2 + g_{vv}v_2^2 = -g_{uv}(u_2^2 - 2u_2v_2 + v_2^2) = -g_{uv}(u_2 - v_2)^2.$$

This shows $G''(t) > (<)0$ iff $g_{uv} < (>)0$ unless $u_2 = v_2$, where

$$g_{uv} = \frac{\partial g_u}{\partial v} = \frac{\partial g_u}{\partial y}\frac{\mathrm{d}\, y}{\partial v} = \left(xf_x(x,y)\right)_y e^v = xyf_{xy}. \tag{3.23}$$

This proof is completed. ∎

Similar to proofs of Corollary 1 or 2, we can get the following Corollary, of which processes are omitted.

Corollary 7. *Suppose $f : \mathbb{D}(\subseteq \mathbb{R}_+ \times \mathbb{R}_+) \to \mathbb{R}$ is a zero-order homogeneous and two-time differentiable function. For given $p, q > 0$ with $p + q = 1$ and arbitrary pairs $(x_1, y_1), (x_2, y_2), (x_1^p x_2^q, y_1^p y_2^q) \in \mathbb{D}$, the following inequality*

$$f(x_1^p x_2^q, y_1^p y_2^q) \le pf(x_1, y_1) + qf(x_2, y_2) \tag{3.24}$$

holds iff $f_{xy} < 0$. With equality iff $x_1 : x_2 = y_1 : y_2$.

By Corollary 3 and 8, we obtain immediately

Corollary 8. *Let $f : \mathbb{D}(\subseteq \mathbb{R}_+ \times \mathbb{R}_+) \to \mathbb{R}_+$ is a zero-order homogeneous and two-time differentiable function. Then for given $p, q > 0$ with $p + q = 1$ and arbitrary pairs $(x_1, y_1), (x_2, y_2), (x_1^p x_2^q, y_1^p y_2^q) \in \mathbb{D}$, the following inequalities*

$$f^p(x_1, y_1)f^q(x_2, y_2) \le f(x_1^p x_2^q, y_1^p y_2^q) \le pf(x_1, y_1) + qf(x_2, y_2) \tag{3.25}$$

hold if $f_{xy} < 0$ and $I = (\ln f)_{xy} > 0$. With equality iff $x_1 : x_2 = y_1 : y_2$.

4. Applications

It sees from theorems and corollaries in section above that for a homogeneous functions we can obtain certain classical or new inequalities by decision signs of f_{xy} and $\acute{I} = (\ln f)_{xy}$.

Example 1 (The Usual Minkowski, Hölder and Chebyshev Inequality)**.** *1) Let $f(x, y) = x + y$. Since $f(x, y)$ is a one-order homogeneous function, by a simple calculation, we get $(\ln f)_{xy} = -\frac{1}{(x+y)^2} < 0$. Applying Corollary 3 and 6,, we obtain the usual Hölder and Chebyshev inequality as follows:*

$$x_1^p x_2^q + y_1^p y_2^q \le (x_1 + y_1)^p (x_2 + y_2)^q \tag{4.1}$$

$$2(x_1x_2 + y_1y_2) \le (x_1 + y_1)(x_2 + y_2) \tag{4.2}$$

2) Let $f(x, y) = (x^p + y^p)^{\frac{1}{p}}$. Since $f(x, y)$ is a one-order homogeneous function, by a simple calculation, we get

$$f_{xy} = (1 - p)(x^p + y^p)^{\frac{1}{p}-2} x^{p-1} y^{p-1},$$

which shows $f_{xy} < 0$ if $p > 1$ and $f_{xy} > 0$ if $p < 1$. Applying corollary 1, we obtain the usual Minkowski inequality as follows:

$$\left[(x_1 + x_2)^p + (y_1 + y_2)^p\right]^{\frac{1}{p}} \le (x_1^p + y_1^p)^{\frac{1}{p}} + (x_2^p + y_2^p)^{\frac{1}{p}} \tag{4.3}$$

for $p > 1$. It is reversed if $p < 1$.

Example 2 (R.Bellman Inequality [3])**.** *If $\mathbf{a} = (a_1, \cdots, a_n)$ and $\mathbf{b} = (b_1, \cdots, b_n)$ are nonnegative real sequences satisfying*

$$a_1^p - \sum_{k=2}^{n} a_k^p > 0 \text{ and } b_1^p - \sum_{k=2}^{n} b_k^p > 0, \tag{4.4}$$

then for $p>1$ there is

$$(4.5)\qquad (a_1^p-\sum_{k=2}^{n}a_k^p)^{\frac{1}{p}}+(b_1^p-\sum_{k=2}^{n}b_k^p)^{\frac{1}{p}}\le\left[(a_1+b_1)^p-\sum_{k=2}^{n}(a_k+b_k)^p\right]^{\frac{1}{p}}.$$

In fact, taking $f(x,y)=(x^p-y^p)^{\frac{1}{p}}$ with $p\neq 0$ and $x>y>0$, $x^p-y^p>0$, then $f_{xy}=(p-1)x^{p-1}y^{p-1}(x^p-y^p)^{\frac{1}{p}-2}$. By Corollary 1, if $f_{xy}>0$ i.e. $p>1$ then

$$(4.6)\qquad (a_1^p-a_2^p)^{\frac{1}{p}}+(b_1^p-b_2^p)^{\frac{1}{p}}\le\left[(a_1+b_1)^p-(a_2+b_2)^p\right]^{\frac{1}{p}}.$$

Based on (4.6), applying mathematical induction, it is easy to obtain (4.5). With equality iff a_k are proportional to $b_k(k=1,2,\cdots,n)$.

Remark 5. *If $p<1, p\neq 0$ then $f_{xy}<0$, from Corollary 1 it follows that R. Bellman inequality is reversed. With equality iff a_k are proportional to $b_k(k=1,2,\cdots,n)$.*

Example 3 (M-type, H-type and Ch-type inequality for Generalized Logarithemic mean)**.** *Let $f(x,y)=S(\alpha;x,y)$ that is defined by (1.5). Since $S(\alpha;x,y)$ is a one-order homogeneous function with respect to x and y, for getting M-type, H-type inequality we need only to confirm the signs of S_{xy} and $(\ln S)_{xy}$ by Corollary 1 and 5.*

For $\alpha\neq 0,1$, by derivative calculations we get

$$(4.7)\qquad I=(\ln S)_{xy}=\frac{1}{(\alpha-1)xy(x-y)^2}\left[x^\alpha y^\alpha\left(\frac{x^\alpha-y^\alpha}{\alpha(x-y)}\right)^{-2}-xy\right]$$
$$=-\frac{1}{xy(x-y)^2G(\alpha)G(1)}\frac{G(\alpha)-G(1)}{\alpha-1}$$

$$(4.8)\qquad \frac{S_{xy}}{S}=-\frac{\alpha x^\alpha y^\alpha}{(x^\alpha-y^\alpha)^2}\left[\left(\frac{x^{\alpha-1}-y^{\alpha-1}}{(\alpha-1)(x-y)}\right)^2x^{1-\alpha}y^{1-\alpha}-x^{-1}y^{-1}\right]$$
$$=-\frac{\alpha^2x^\alpha y^\alpha}{(x^\alpha-y^\alpha)^2}\frac{G(\alpha-1)-G(-1)}{\alpha-1-(-1)},$$

where $G(t)=\left[\frac{x^t-y^t}{t(x-y)}\right]^2x^{-t}y^{-t}$ $(t\neq 0)$, $G(0)\overset{\triangle}{=}\lim_{t\to 0}G(t)=L^{-2}(x,y)$.

Obviously, $G(t)$ satisfies that $G(t)=G(-t)$ and is strictly increasing in t on $(0,+\infty)$ because

$$\frac{G'(t)}{G(t)}=\frac{2(x^t\ln x-y^t\ln y)}{x^t-y^t}-\frac{2}{t}-\ln xy$$
$$=\frac{2}{t}\left[\frac{x^t\ln x^t-y^t\ln y^t}{x^t-y^t}-1-\ln\sqrt{x^ty^t}\right]$$
$$=\frac{2}{t}\left(\ln E(x^t,y^t)-\ln\sqrt{x^ty^t}\right)$$
$$=\begin{cases}>0 & if\ t>0\\ <0 & if\ t<0\end{cases}.$$

So

$$\text{(4.9)} \quad \operatorname{sgn}\frac{G(t_2)-G(t_1)}{t_2-t_1} = \operatorname{sgn}\frac{G(|t_2|)-G(|t_1|)}{t_2-t_1} = \operatorname{sgn}\left[\frac{|t_2|-|t_1|}{t_2-t_1}\frac{G(|t_2|)-G(|t_1|)}{|t_2|-|t_1|}\right] = \operatorname{sgn}\frac{|t_2|-|t_1|}{t_2-t_1} = \operatorname{sgn}(t_2+t_1).$$

Thus we have

$$\text{(4.10)} \quad \operatorname{sgn} I = -\operatorname{sgn}\frac{G(\alpha)-G(1)}{\alpha-1} = \begin{cases} 1 & \text{if } \alpha < -1 \\ -1 & \text{if } \alpha > -1 \end{cases},$$

$$\text{(4.11)} \quad \operatorname{sgn}\frac{S_{xy}}{S} = -\operatorname{sgn}\frac{G(\alpha-1)-G(-1)}{\alpha-1-(-1)} = \begin{cases} 1 & \text{if } \alpha < 2 \\ -1 & \text{if } \alpha > 2 \end{cases}.$$

Using Corollary 1 and 5, we obtain

$$\text{(4.12)} \quad S(\alpha; x_1^p x_2^q, y_1^p y_2^q) \le S^p(\alpha; x_1, y_1)\cdot S^q(\alpha; x_2, y_2) \text{ if } \alpha > -1,$$

$$\text{(4.13)} \quad S(\alpha; x_1x_2, y_1y_2) \le S(\alpha; x_1, y_1)\cdot S(\alpha; x_2, y_2) \quad \text{if } \alpha > -1,$$

$$\text{(4.14)} \quad S(\alpha; px_1+qx_2, py_1+qy_2) \le pS(\alpha; x_1, y_1) + qS(\alpha; x_2, y_2) \text{ if } \alpha > 2,$$

where $pq > 0$ with $p + q = 1$. With equality iff $x_1 : x_2 = y_1 : y_2$. For $\alpha < -1$ and $\alpha < 2$, (4.12), (4.13) and (4.14) are reversed respectively.

Likewise for $\alpha = 0, 1$ we can get corresponding inequalities.

Example 4 (M-type, H-type and Ch-type Inequality for One-parameter Mean)**.** *Let $f(x,y) = J(\alpha; x, y)$ $(x, y > 0$ with $x \ne y)$, where $J(\alpha; x, y)$ is defined by (1.4). Since $J(\alpha; x, y)$ is a homogeneous function with respect to x and y, for getting M-type, H-type inequality we need only to confirm the signs of J_{xy} and $(\ln J)_{xy}$ by Corollary 1 and 5.*

For $\alpha \ne 0, -1$, by derivative calculations we get

$$\text{(4.15)} \quad I = (\ln J)_{xy} = \frac{1}{xy(x-y)^2}\left[\frac{x^{\alpha+1}y^{\alpha+1}}{\left(\frac{x^{\alpha+1}-y^{\alpha+1}}{(\alpha+1)(x-y)}\right)^2} - \frac{x^\alpha y^\alpha}{\left(\frac{x^\alpha-y^\alpha}{\alpha(x-y)}\right)^2}\right] = -\frac{1}{xy(x-y)^2}\frac{G(\alpha{+}1)-G(\alpha)}{(\alpha+1)-\alpha},$$

$$\text{(4.16)} \quad \frac{J_{xy}}{J} = -\frac{2\alpha(\alpha+1)x^\alpha y^\alpha}{(x^{\alpha+1}-y^{\alpha+1})(x^\alpha-y^\alpha)}\left(\frac{\alpha}{\alpha+1}\frac{x^{\alpha+1}-y^{\alpha+1}}{x^\alpha-y^\alpha} - \frac{x+y}{2}\right) = -\frac{2\alpha(\alpha+1)x^\alpha y^\alpha}{(x^{\alpha+1}-y^{\alpha+1})(x^\alpha-y^\alpha)}[J(\alpha+1)-J(1)].$$

By (4.9) we have

$$\text{(4.17)} \quad \operatorname{sgn} I = -\operatorname{sgn}\frac{G(\alpha{+}1)-G(\alpha)}{(\alpha+1)-\alpha} = \begin{cases} 1 & \text{if } \alpha < -\frac{1}{2} \\ -1 & \text{if } \alpha > -\frac{1}{2} \end{cases}.$$

Since

$$\frac{2\alpha(\alpha+1)x^\alpha y^\alpha}{(x^{\alpha+1}-y^{\alpha+1})(x^\alpha-y^\alpha)} = \frac{2x^\alpha y^\alpha}{(x-y)^2}\frac{(\alpha+1)(x-y)}{x^{\alpha+1}-y^{\alpha+1}}\frac{\alpha(x-y)}{x^\alpha-y^\alpha} > 0$$

and $J(\alpha)$ *is strictly increasing in* α *on* $(-\infty, +\infty)$*, we have*

$$\operatorname{sgn}(J_{xy}/J) = -\operatorname{sgn}[J(\alpha) - J(1)] = \begin{cases} 1 & \text{if } \alpha < 1 \\ -1 & \text{if } \alpha > 1 \end{cases}. \tag{4.18}$$

And then we obtain

$$J(\alpha; x_1^p x_2^q, y_1^p y_2^q) \leq J^p(\alpha; x_1, y_1) \cdot J^q(\alpha; x_2, y_2) \text{ if } \alpha > -\tfrac{1}{2}, \tag{4.19}$$

$$J(\alpha; x_1 x_2, y_1 y_2) \leq J(\alpha; x_1, y_1) \cdot J(\alpha; x_2, y_2) \text{ if } \alpha > -\tfrac{1}{2}, \tag{4.20}$$

$$J(\alpha; px_1 + qx_2, py_1 + qy_2) \leq pJ(\alpha; x_1, y_1) + qJ(\alpha; x_2, y_2) \text{ if } \alpha > 1, \tag{4.21}$$

where $pq > 0$ *with* $p + q = 1$*. With equality iff* $x_1 : x_2 = y_1 : y_2$*. For* $\alpha < -\frac{1}{2}$ *and* $\alpha < 1$*, (4.19), (4.20) and (4.21) are reversed respectively.*

Likewise for $\alpha = 0, -1$*, we can get corresponding inequalities.*

Example 5. *Let* $f(x, y) = \frac{x-y}{y}, (x, y) \in \mathbb{R}_+ \times \mathbb{R}_+, x > y$*. Then* $f(x, y)$ *is a zero-order homogeneous function and satisfies* $f_{xy} = -\frac{1}{y^2} < 0, (\ln f)_{xy} = \frac{1}{(x-y)^2} > 0$*. By Corollary 8, for* $x_1 > y_1 > 0, x_2 > y_2 > 0, pq > 0$ *with* $p + q = 1$ *we have*

$$\left(\frac{x_1 - y_1}{y_1}\right)^p \left(\frac{x_2 - y_2}{y_2}\right)^q \leq \frac{x_1^p x_2^q - y_1^p y_2^q}{y_1^p y_2^q} \leq p\frac{x_1 - y_1}{y_1} + q\frac{x_2 - y_2}{y_2}. \tag{4.22}$$

Put $\frac{x_1 - y_1}{y_1} = a_1, \frac{x_2 - y_2}{y_2} = a_2$*, then inequalities above can be rewritten as*

$$a_1^p a_2^q \leq (1 + a_1)^p (1 + a_2)^q - 1 \leq pa_1 + qa_2, \tag{4.23}$$

which can be generalized as

$$a_1^{p_1} \cdots a_n^{p_n} \leq (1 + a_1)^{p_1} \cdots (1 + a_n)^{p_n} - 1 \leq p_1 a_1 + \cdots + p_n a_n, \tag{4.24}$$

where $a_i, p_i > 0$ $(i = 1, 2, \cdots, n)$ *with* $\sum_{i=1}^n p_i = 1$*. With equality iff all* a_i *are equal.*

Remark 6. *Inequalities (4.24) shows* $\prod_{i=1}^n (1 + a_i)^{p_i} - 1$ *is a mean value of positive numbers* a_i *between the geometric mean* $\prod_{i=1}^n a_i^{p_i}$ *and arithmetic mean* $\sum_{i=1}^n p_i a_i$*.*

Applying theorems and corollaries in this paper, we can prove further other H-type, Ch-type and M-type inequalities for homogeneous functions, which will be discussed another paper.

References

[1] H. Alzer, Üer eine einparametrige familie von Mitlewerten, *Bayer. Akad. Wiss. Math.–Natur. Kl. Sitzungsber.* **1987**(1988), 23–29. (German)

[2] H. Alzer, Üer eine einparametrige familie von Mitlewerten, II, *Bayer. Akad. Wiss. Math.–Natur. Kl. Sitzungsber.* **1988**(1989), 23–29. (German)

[3] R. Bellman, On an Inequality Concerning an Indefinite Form, *Amer. Math. Monthly*, **63**(1956), 108-109.

[4] G. H. Hardy, J. E. Littlewood and G. Pòlya, *Inequalities*, *2nd Ed.* Cambridge University Press, 1952.

[5] 24 L. Losonczi AND Zs. Páles, Minkowski's inequality for two variable difference means, *Proc. Amer. Math. Soc.* **126**(1998), 779-789.

[6] Hongwei Lou, Hlder inequalities of means, *J. of Ningbo Univ.*, **9**(1)(1996), 1-7.

[7] Constantin P. Niculescu, Convexity According to the Geometric Mean, *Mathematical Inequalities & Applications*, **3**(2)(2000), 155-167.

[8] J. Sándor, On Lehman's inequality and electrical networks, *RGMIA Research Report Collection*, **8**(1), Article 1, 2005.
[9] Zhen-Hang Yang, Holder,Minkowski, and Tchebychef Type Inequalities for Logarithmic Mean and Exponential Mean, *J. of Xuzhou Normal Univ. (Natural Science Edition)*, **23**(1)(2005), 31-34.
[10] Zhen-Hang Yang, On the Homogeneous Functions with Two Parameters and Its Monotonicity, *RGMIA Research Report Collection*, **8**(2), Article 10, 2005. [ONLINE] Available online at `http://rgmia.vu.edu.au/v8n2.html.`
[11] Xiao-Min Zhang, *Geometrically Convex Function*, An'hui University Press, 2004. (Chinese).
[12] Xiao-Min Zhang, Some Theorem on Geometric Convex Function and Its Applictions, *Journal of Capital Normal Universit*, **25**(2)(2004), 11-13. (Chinese).

Zhejiang Electric Power Vocational Technical College, Hangzhou, Zhejiang, China, 311600
E-mail address: `yzhkm@163.com`