On the monotonicity of the sequence (σ_k/σ_k^*)

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1 Introduction

Let $n > 1$ be a positive integer and $k \geq 0$ a nonnegative integer. A divisor d of n is called a unitary divisor of n, if $(d, n/d) = 1$. Let $\sigma^*(n)$ be the sum of unitary divisors of n, i.e.

$$\sigma^*(n) = \sum_{d|n,(d,n/d)=1} d. \tag{1}$$

Then it is well-known (see e.g. [1], [8]) that

$$\sigma^*(n) = \prod_{i=1}^{r} (p_i^{a_i} + 1), \tag{2}$$

where $n = \prod_{i=1}^{n} p_i^{a_i}$ is the prime factorization of $n > 1$ (p_i distinct primes, $a_i \geq 1$ positive integers). More generally, if $\sigma_k^*(n)$ is the sum of kth powers of unitary divisors of n (i.e. (1) generalized to d^k in place of d in the sum), then, similarly to (2), one has

$$\sigma_k^*(n) = \prod_{i=1}^{r} (p_i^{ka_i} + 1). \tag{3}$$

We note that, for $k = 0$ we get the number $d^*(n) = \sigma_0^*(n)$ of unitary divisors of n, when (3) gives

$$d^*(n) = 2^r = 2^{\omega(n)}, \tag{4}$$

where $\omega(n) = r$ denotes the number of unitary divisors of n. The similar formulae for the (classical) sum of divisors of n are the well-known (see e.g. [2], [9], [7])

$$\sigma(n) = \prod_{i=1}^{r} (p_i^{a_i+1} - 1)/(p_i - 1), \tag{5}$$

resp.

$$\sigma_k(n) = \prod_{i=1}^{r} (p_i^{k(a_i+1)} - 1)/(p_i^k - 1). \tag{6}$$

For $k = 0$, (6) provides the number $d(n)$ of classical divisors of n:

$$d(n) = \prod_{i=1}^{r} (a_i + 1). \tag{7}$$

There are many results involving inequalities on these arithmetical functions. See e.g. [3]-[6]. For surveys of results, see e.g. [9], [8].

2 Main results

Langford ([9]) proved that

$$\sigma_k(n) \leq d(n) \left(\frac{n^k + 1}{2} \right), \tag{8}$$

while we proved ([10], [4], [5]) the stronger relation

$$\sigma_k(n) \leq \frac{d(n)\sigma_k^*(n)}{2^{\omega(n)}} \leq d(n) \left(\frac{n^k + 1}{2} \right). \tag{9}$$

The second inequality of (9) is a consequence of the elementary inequality

$$\prod_{i=1}^{r}(x_i+1) \le 2^{r-1}\left(\prod_{i=1}^{r} x_i + 1\right) \quad (x_i \ge 1,\ r \ge 1) \tag{10}$$

applied to $x_i = p_i^{ka_i}$, $r = \omega(n)$, and using relation (3).

Remark that the first inequality of (9) may be written also as

$$\frac{\sigma_k(n)}{\sigma_k^*(n)} \le \frac{\sigma_0(n)}{\sigma_0^*(n)}. \tag{11}$$

Our aim is to give a generalization of (11) as follows:

Theorem. *For all fixed $n \ge 1$, the sequence $\left(\dfrac{\sigma_k(n)}{\sigma_k^*(n)}\right)_{k\ge 0}$ is monotone decreasing.*

Proof. We have to prove that

$$\frac{\sigma_k(n)}{\sigma_k^*(n)} \le \frac{\sigma_l(n)}{\sigma_l^*(n)} \text{ for all } k \ge l \ge 0. \tag{12}$$

By (3) and (6),

$$f_k(n) = \sigma_k(n)/\sigma_k^*(n) = \prod_{i=1}^{r}(p_i^{k(a_i+1)}-1)/(p_i^k-1)(p_i^{ka_i}+1),$$

so to prove that $f_k(n) \le f_l(n)$ for $k \ge l$, it will be sufficient to show that

$$\frac{p^{k(a+1)}-1}{(p^k-1)(p^{ka}+1)} \le \frac{p^{l(a+1)}-1}{(p^l-1)(p^{la}+1)}, \quad k \ge l \ge 0,\ p \ge 2. \tag{13}$$

Put $p^k = x$, $p^l = y$, where $x > y \ge 1$. After some elementary transformations (which we omit here) it can be shown that (13) becomes equivalent to

$$\frac{x^a - y^a}{x-y} \le \frac{(xy)^a - 1}{xy - 1} \quad (x > y \ge 1). \tag{14}$$

For $y = 1$, relation (14) is trivial, so we may suppose $y \geq 2$. Now, remark that

$$\frac{x^a - y^a}{x - y} = x^{a-1} + x^{a-2}y + \cdots + xy^{a-2} + y^{a-1} \leq ax^{a-1},$$

by $y < x$ and $a \geq 1$. On the other hand, we will prove that

$$\frac{(xy)^a - 1}{xy - 1} \geq ax^{a-1}. \tag{15}$$

This is equivalent to

$$(xy)^a - 1 \geq ax^a y - ax^{a-1},$$

or

$$x^a y(y^{a-1} - a) + ax^{a-1} - 1 \geq 0.$$

Here $ax^{a-1} - 1 \geq a - 1 \geq 0$, and $y^{a-1} - a \geq 2^{a-1} - a \geq 0$ for all $a \geq 1$, so the result follows. By (15), and the above remark, inequality (14) is established. By (13), the inequality (12) follows, so the theorem is proved.

Remarks. 1) For $l = 0$, $k \geq 0$ arbitrary, we reobtain relation (11).

2) For $l = 1$, $k \geq 1$ we get

$$\frac{\sigma_k(n)}{\sigma_k^*(n)} \leq \frac{\sigma(n)}{\sigma^*(n)}, \tag{16}$$

which offers an improvement of (11) for $k \geq 1$, since by (11) applied to $k = 1$, and by (16), one has

$$\frac{\sigma_k(n)}{\sigma_k^*(n)} \leq \frac{\sigma(n)}{\sigma^*(n)} \leq \frac{\sigma_0(n)}{\sigma_0^*(n)} = \frac{d(n)}{d^*(n)}. \tag{17}$$

For other improvements of the right side of (17), see [5].

References

[1] E. Cohen, *Arithmetical functions associated with the unitary divisors of an integer*, Math. Z. **74**(1960), 66-80.

[2] G. H. Hardy and E. M. Wright, *An introduction to the theory of numbers*, Oxford Univ. Press, 1960.

[3] J. Sándor, *An application of the Jensen-Hadamard inequality*, Nieuw Arch. Wiskunde, Serie 4, **8**(1990), no. 1, 43-66.

[4] J. Sándor, *On an inequality of Klamkin with arithmetical applications*, Int. J. Math. Ed. Sci. Technol. **25**(1994), 157-158.

[5] J. Sándor, *On certain inequalities for arithmetic functions*, Notes Numb. Th. Discr. Math. **1**(1995), 27-32.

[6] J. Sándor, *On the arithmetical functions $d_k(n)$ and $d_k^*(n)$*, Portugaliae Math. **53**(1996), no. 1, 107-115.

[7] J. Sándor, *Geometric theorems, diophantine equations, and arithmetic functions*, American Research Press, Rehoboth, New Mexico, 2002.

[8] J. Sándor, *Handbook of number theory II*, Springer-Verlag, 2004.

[9] J. Sándor, D. S. Mitrinović, *Handbook of number theory*, Kluwer Acad. Publ., 1996.

[10] J. Sándor, L. Tóth, *On certain number-theoretic inequalities*, Fib. Quart. **28**(1990), no. 3, 255-258.