On an inequality of Kodokostas

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Abstract. We obtain a proof, based on one of the Hadamard integral inequalities, of an improved form for an inequality by D. Kodokostas [1].

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1 Introduction

In a recent note, D. Kodokostas [1] has proved two inequalities, where the second inequality follows at once from the first one, stated as:

Theorem 1. *If x, y are positive real numbers, and $n \geq 2$ is a positive integer, then*

$$(x^n + y^n) / \left(\sum_{i=1}^{n} x^{n-i} y^i \right) \geq \frac{x+y}{n}. \tag{1}$$

Since (for $x \neq y$) by the algebraic identity $\sum_{i=1}^{n} x^{n-i} y^i = \dfrac{x^n - y^n}{x - y}$, relation (1) may be formulated equivalently as

$$\frac{x^n + y^n}{x + y} \geq \frac{1}{n} \cdot \frac{x^n - y^n}{x - y}, \tag{2}$$

we could try to evaluate the right side of (2) by using the integral $\int_x^y t^{n-1} dt$. We will prove that (2) holds true for any real number $n \geq 2$. Our method is based on the right side of the classical Hadamard inequality ([3], [2]).

Lemma. *Let $f : [a, b] \to \mathbb{R}$ be a continuous convex function. Then one has the double inequality*

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(t)dt \leq \frac{f(a) + f(b)}{2}. \tag{3}$$

2 Main results

Theorem 2. *Let $\lambda \geq 2$ be a real number. Then*

$$\left(\frac{x+y}{2}\right)^{\lambda-1} \leq \frac{1}{\lambda} \cdot \frac{x^\lambda - y^\lambda}{x - y} \leq \frac{x^{\lambda-1} + y^{\lambda-1}}{2} \leq \frac{x^\lambda + y^\lambda}{x + y} \tag{4}$$

for any positive real numbers $x \neq y$.

Proof. Put $a = y < x = b$, $f(t) = t^{\lambda-1}$ in (3). Since

$$f''(t) = (\lambda - 1)(\lambda - 2)t^{\lambda-3} \geq 0,$$

f is convex on $[a, b]$. By using the Lemma, Theorem 2 follows, excepting the last inequality of (4). This is equivalent to $xy^{\lambda-1} + yx^{\lambda-1} \leq x^\lambda + y^\lambda$, or $(x^{\lambda-1} - y^{\lambda-1})(x-y) \geq 0$ which holds true. Since $\lambda \geq 2$ and $x > y$, here we cannot have an equality. However, in the stronger form of (4), there is equality only for $\lambda = 2$ (otherwise f is strictly convex, and in (3) one has strict inequalities, see e.g. [2]).

Remarks. For $\lambda = n$ the second and fourth terms in Theorem 2 give Kodokostas' inequality (2). We cannot have equality, but in the original relation (1) this happens for $x = y$.

For extensions and generalizations or refinements of the Hadamard integral inequalities see e.g. [3], [2], and the references therein.

References

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