INTEGRAL INEQUALITIES INVOLVING RANDOM VARIABLES

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Abstract. Probability density functions of random variables jointly with classical inequalities for convex functions are used and some integral inequalities are obtained.

1. Introduction

Inequalities are an essential part of virtually almost all branches of mathematics and their applications as in other areas of science. This is reflected in the fact that they have attracted the attention of many mathematicians and in the extensive literature that exist on the subject. Classical Jensen's inequality ([1], [2], [3]) is our starting point in this paper where properties of well known random variables [4] are used to obtain several integral inequalities. We have applied a technique similar to the one used in ([5], [6], [7]) where some elementary discrete inequalities were obtained.

2. The Inequalities

In what follows we present some bounds for integrals that can be obtained using the probability density function and the moments of random variables. We start with the following theorem.

Theorem 2.1. *Let X be a random variable with probability density function $h_X(x) = h(x)I\{a \le x \le b\}, 0 \le a \le b$. Then, for all $r \in \mathbb{N}$ holds*

$$\frac{1}{1+E(X^r)} \le \int_a^b \frac{h(x)}{1+x^r}\, dx \tag{2.1}$$

where $E(X^r)$ represents the r^{th} moment of X.

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Proof. To proof the preceding statement we will apply Jensen's inequality for integrals. Namely, if $f : [a,b] \to \mathbb{R}$ is a convex function and $h : [a,b] \to \mathbb{R}_+^*$ and $u : [a,b] \to \mathbb{R}_+$ are integrable functions, then

$$f\left(\frac{\int_a^b h(x)u(x)\,dx}{\int_a^b h(x)\,dx}\right) \le \frac{\int_a^b h(x)f(u(x))\,dx}{\int_a^b h(x)u(x)\,dx} \tag{2.2}$$

Setting $f : [a,b] \to \mathbb{R}, (0 \le a < b)$, defined by $f(t) = \frac{1}{1+t}$, that is convex in $(-1,+\infty)$, and $h(t) = h_X(t)$ and $u(t) = t^r, r \in \mathbb{N}$ into (2.2), we have

$$\left(1 + \frac{\int_a^b t^r h(t)\,dt}{\int_a^b h(t)\,dt}\right)^{-1} \le \frac{\int_a^b \frac{h(t)}{1+t^r}\,dt}{\int_a^b h(t)\,dt}$$

From the preceding expression and taking into account that the r^{th} moment of X is given by $E(X^r) = \int_a^b t^r h(t)\,dt$ and the fact that $\int_a^b h(t)\,dt = 1$, the statement immediately follows and this completes the proof.

□

Notice that when $r = 1$ then we get the following lower bound for the expectation of X :

$$E(X) \ge \frac{1 - \displaystyle\int_a^b \frac{h(t)}{1+t}\,dt}{\displaystyle\int_a^b \frac{h(t)}{1+t}\,dt}$$

An immediate consequence of Theorem 2.1 is the following

Corollary 2.2. *Let λ be a positive real number. Then, holds*

$$\frac{1}{1+\lambda} \le \int_0^\infty \frac{e^{-\lambda t}}{1+t}\,dt$$

Proof. Setting $r = 1$, $f(t) = \frac{1}{1+t}$ and $h_X(t) = \lambda e^{-\lambda t}$, the probability density function of an exponential random variable, into (2.1) and taking into account that $E(X) = 1/\lambda$, we get

$$\frac{1}{1+1/\lambda} \le \int_0^\infty \frac{\lambda e^{-\lambda t}}{1+t}\,dt.$$

Rearranging terms and after simplification the claimed inequality immediately follows and the proof is complete.

□

Using the same procedure with probability density functions of another random variables several bounds for integrals are obtained. They are given in the following corollaries.

Corollary 2.3. *Let α be a positive real number. Then, the following inequality*

$$\frac{1}{2\alpha+\sqrt{\alpha\pi}} \le \int_0^{\infty} \frac{te^{-\alpha t^2}}{1+t}\,dt$$

holds.

Proof. Let X be a Rayleigh's random variable which probability density function $h_X(t)$ is given by $h_X(t) = 2\alpha te^{-\alpha t^2} I\{0 < t < +\infty\}$ with expectation $E(X) = \frac{1}{2}\sqrt{\frac{\pi}{\alpha}}$. Then, applying (2.1), we get

$$\frac{1}{1+\frac{1}{2}\sqrt{\frac{\pi}{\alpha}}} \le \int_0^{\infty} \frac{2\alpha te^{-\alpha t^2}}{1+t}\,dt$$

from which the statement immediately follows and this completes the proof.

□

Corollary 2.4. *Let $\alpha \neq 1$ be a positive real number. Then, the following inequality*

$$\frac{\alpha-1}{\alpha(2\alpha-1)} \le \int_1^{\infty} \frac{1}{t^{\alpha+1}(t+1)}\,dt$$

holds.

Proof. Let X be a Pareto's random variable which probability density function is $h_X(t) = \frac{\alpha}{t^{\alpha+1}} I\{1 < t < +\infty\}$. Since its expectation is given by $E(X) = \frac{\alpha}{\alpha-1}$, then applying (2.1), we get

$$\frac{1}{1+\frac{\alpha}{\alpha-1}} \le \int_1^{\infty} \frac{\alpha}{t^{\alpha+1}(t+1)}\,dt$$

and this completes the proof.

□

Corollary 2.5. *Let a, b be positive real numbers. Then, holds*

$$\int_0^1 t^{a-1}(1-t)^{b-1}\log(t)\,dt \le \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}\log\left(\frac{a}{a+b}\right)$$

where $\Gamma(x)$ is the gamma function.

Proof. Let X be a Beta's random variable which probability density function is

$$\begin{aligned} h_X(t) &= \frac{1}{B(a,b)} t^{a-1}(1-t)^{b-1} I\{0<t<1\} \\ &= \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} t^{a-1}(1-t)^{b-1} I\{0<t<1\} \end{aligned}$$

with expectation given by $E(X) = \frac{a}{a+b}$. Setting $u(t) = t$ and $f(t) = \log(t)$, that is concave in $(0,1)$, into (2.2), we have

$$\begin{aligned} \log\left(\frac{a}{a+b}\right) &\geq \frac{1}{B(a,b)} \int_0^1 t^{a-1}(1-t)^{b-1} \log(t)\, dt \\ &= \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \int_0^1 t^{a-1}(1-t)^{b-1} \log(t)\, dt \end{aligned}$$

and the proof is complete.

□

Finally, we give similar results to the preceding ones for centered moments of random variables. We begin with

Theorem 2.6. *Let X be a random variable with probability density function $h_X(x) = h(x)I\{a \leq x \leq b\}, 0 \leq a \leq b$. Then, for all $r \in \mathbb{N}$ holds*

$$\frac{1}{1 + E(X - E(X))^r} \leq \int_a^b \frac{h(x)}{1 + (x - E(X))^r}\, dx \tag{2.3}$$

where $E(X - E(X))^r$ represents the r^{th} centered moment of X.

Proof. To proof (2.3) we will argue in the same way as we have done in the proof of theorem 2.1 setting $u(t) = (t - E(X))^r$.

□

Notice that when $r = 2$, we have

$$Var(X) \geq \frac{1 - \displaystyle\int_a^b \frac{h(t)}{1 + (t - E(X))^2}\, dt}{\displaystyle\int_a^b \frac{h(t)}{1 + (t - E(X))^2}\, dt}$$

An immediate consequence of Theorem 2.6 is the following corollary.

Corollary 2.7. *Let λ be a positive real number. Then, holds*

$$\frac{1}{\lambda(1+\lambda^2)} \leq \int_0^\infty \frac{e^{-\lambda t}}{\lambda^2 + (\lambda t - 1)^2}\, dt$$

Proof. Setting $r = 2$, $f(t) = \frac{1}{1+t}$ and $h_X(t) = \lambda e^{-\lambda t}$, the probability density function of an exponential random variable, into (2.3) and taking into account that $E(X) = 1/\lambda$ and $Var(X) = 1/\lambda^2$ the result immediately follows.

□

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