

APPROXIMATION OF THE DILOGARITHM FUNCTION

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ABSTRACT. In this short note, we approximate Dilogarithm function, defined by $\text{dilog}(x) = \int_1^x \frac{\log t}{1-t} dt$. Letting

$$\mathcal{D}(x, N) = -\frac{1}{2} \log^2 x - \frac{\pi^2}{6} + \sum_{n=1}^N \frac{\frac{1}{n^2} + \frac{1}{n} \log x}{x^n},$$

we show that for every $x > 1$, the inequalities

$$\mathcal{D}(x, N) < \text{dilog}(x) < \mathcal{D}(x, N) + \frac{1}{x^N}$$

holds true for all $N \in \mathbb{N}$.

Definition. The Dilogarithm function $\text{dilog}(x)$ is defined for every $x > 0$ as follows [5]:

$$\text{dilog}(x) = \int_1^x \frac{\log t}{1-t} dt.$$

Expansion. The following expansion holds true when x tends to infinity:

$$\text{dilog}(x) = \mathcal{D}(x, N) + O\left(\frac{1}{x^{N+1}}\right),$$

where

$$\mathcal{D}(x, N) = -\frac{1}{2} \log^2 x - \frac{\pi^2}{6} + \sum_{n=1}^N \frac{\frac{1}{n^2} + \frac{1}{n} \log x}{x^n}.$$

Aim of Present Work. The aim of this note is to prove that:

$$0 < \text{dilog}(x) - \mathcal{D}(x, N) < \frac{1}{x^N} \quad (x > 1, N \in \mathbb{N}).$$

Lower Bound. For every $x > 0$ and $N \in \mathbb{N}$, let:

$$\mathcal{L}(x, N) = \text{dilog}(x) - \mathcal{D}(x, N).$$

A simple computation, yields that:

$$\frac{d}{dx} \mathcal{L}(x, N) = \log x \left(\frac{x}{1-x} + \sum_{n=0}^{N+1} \frac{1}{x^n} \right) < \log x \left(\frac{x}{1-x} + \sum_{n=0}^{\infty} \frac{1}{x^n} \right) = 0.$$

So, $\mathcal{L}(x, N)$ is a strictly decreasing function of the variable x , for every $N \in \mathbb{N}$. Considering $\mathcal{L}(x, N) = O\left(\frac{1}{x^{N+1}}\right)$, we obtain desired lower bound for the Dilogarithm function, as follow:

$$\mathcal{L}(x, N) > \lim_{x \rightarrow +\infty} \mathcal{L}(x, N) = 0.$$

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Upper Bound. For every $x > 0$ and $N \in \mathbb{N}$, let:

$$\mathcal{U}(x, N) = \text{dilog}(x) - \mathcal{D}(x, N) - \frac{1}{x^N}.$$

First, we observe that

$$\mathcal{U}(1, N) = \frac{\pi^2}{6} - \sum_{n=1}^N \frac{1}{n^2} - 1 = \Psi(1, N+1) - 1 \leq \frac{\pi^2}{6} - 2 < 0,$$

in which $\Psi(m, x)$ is the m -th polygamma function, the m -th derivative of the digamma function, $\Psi(x) = \frac{d}{dx} \log \Gamma(x)$, with $\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt$ (see [1, 2]). A simple computation, yields that:

$$\frac{d}{dx} \mathcal{U}(x, N) = \log x \left(\frac{x}{1-x} + \sum_{n=0}^{N+1} \frac{1}{x^n} \right) + \frac{N}{x^{N+1}}.$$

To determine the sign of $\frac{d}{dx} \mathcal{U}(x, N)$, we distinguish two cases:

◇ Suppose $x > 1$. Since, $\frac{\log x}{x-1}$ is strictly decreasing, we have

$$N \geq 1 = \lim_{x \rightarrow 1} \frac{\log x}{x-1} > \frac{\log x}{x-1},$$

which is $\frac{N}{\log x} > \frac{1}{x-1}$ or equivalently $\frac{N}{x^{N+1} \log x} > \sum_{n=N+2}^{\infty} \frac{1}{x^n}$, and this yields that $\frac{d}{dx} \mathcal{U}(x, N) > 0$. So, $\mathcal{U}(x, N)$ is strictly increasing for every $N \in \mathbb{N}$. Thus, $\mathcal{U}(x, N) < \lim_{x \rightarrow +\infty} \mathcal{U}(x, N) = 0$; as desired in this case. Also, note that in this case we obtain

$$\mathcal{U}(x, N) > \mathcal{U}(1, N) = \Psi(1, N+1) - 1.$$

◇ Suppose $0 < x < 1$ and $N - \frac{\log x}{x-1} \geq 0$. We observe that $1 < \frac{\log x}{x-1} < +\infty$ and $\sum_{n=0}^{N+1} \frac{1}{x^n} = \frac{1-x^{N+2}}{x^{N+1}(1-x)}$. Considering these facts, we see that $\frac{d}{dx} \mathcal{U}(x, N)$ and $N - \frac{\log x}{x-1}$ have same sign; i.e.

$$\text{sgn} \left(\frac{d}{dx} \mathcal{U}(x, N) \right) = \text{sgn} \left(N - \frac{\log x}{x-1} \right).$$

Thus, $\mathcal{U}(x, N)$ is increasing and so,

$$\mathcal{U}(x, N) \leq \lim_{x \rightarrow 1^-} \mathcal{U}(x, N) = \Psi(1, N+1) - 1 \leq \frac{\pi^2}{6} - 2 < 0.$$

Connection with Other Functions. Using Maple software, we have:

$$\begin{aligned} \mathcal{D}(x, N) &= -\frac{1}{2} \log^2 x - \frac{\pi^2}{6} + \frac{1}{N^2 x^N} + \frac{\log x}{N x^N} - \log \left(\frac{x-1}{x} \right) \log x \\ &+ \text{polylog} \left(2, \frac{1}{x} \right) - \frac{\log x}{x^N} \Phi \left(\frac{1}{x}, 1, N \right) - \frac{1}{x^N} \Phi \left(\frac{1}{x}, 2, N \right), \end{aligned}$$

in which

$$\text{polylog}(a, z) = \sum_{n=1}^{\infty} \frac{z^n}{n^a},$$

is the polylogarithm function of index a at the point z and defined by above series if $|z| < 1$, and by analytic continuation otherwise [4]. Also,

$$\Phi(z, a, v) = \sum_{n=1}^{\infty} \frac{z^n}{(v+n)^a},$$

is the Lerch zeta (or Lerch- Φ) function defined by above series for $|z| < 1$, with $v \neq 0, -1, -2, \dots$, and by analytic continuation, it is extended to the whole complex z -plane for each value of a and v (see [3, 6]).

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