HÖLDER, CHEBYSHEV AND MINKOWSKI TYPE INEQUALITIES FOR STOLARSKY MEANS

ZHEN-HANG YANG

ABSTRACT. In this paper, we study the AA-convexity and GG-convexity of Stolarsky means, of which Hölder, Chebyshev and Minkowski type inequalities are given, where another new and concise proof of Minkowski type inequality for Stolarsky means is contained.

1. INTRODUCTION

For one-parameter means of two positive numbers x and y denoted by

$$J(\alpha;x,y)=\begin{cases}\frac{\alpha(x^{\alpha+1}-y^{\alpha+1})}{(\alpha+1)(x^{\alpha}-y^{\alpha})}, & \alpha\neq 0,-1, x\neq y;\\ \frac{x-y}{\ln x-\ln y}, & \alpha=0, x\neq y;\\ \frac{xy(\ln x-\ln y)}{x-y}, & \alpha=-1, x\neq y;\\ y & x=y\end{cases} \tag{1.1}$$

A. Horst presented Chebyshev and Minkowski type inequalities in 1988 (see [1, 2].

For generalized logarithmic means of two positive numbers x and y denoted by

$$S(\alpha;x,y)=\begin{cases}\left(\frac{x^{\alpha}-y^{\alpha}}{\alpha(x-y)}\right)^{\frac{1}{\alpha-1}} & \alpha\neq 0,1, x\neq y;\\ L(x,y) & \alpha=0, x\neq y;\\ E(x,y) & \alpha=1, x\neq y;\\ y & x=y.\end{cases} \tag{1.2}$$

Hongwei Lou proved that the H-type inequality of $S(\alpha;x,y)$ is valid in 1996 (see [4]). Zhen-Hang Yang proved that its Hölder, Chebyshev and Minkowski type inequalities are also true by using classical integral inequalities in 2005 (see [6])

Date: October 23, 2005.

2000 *Mathematics Subject Classification.* Primary 26D07,26E60; Secondary 26A48, 26A51.

Key words and phrases. Hölder inequality, Chebyshev inequality, Minkowski inequality, Stolarsky means.

The so-called Stolarsky means (extended means) of positive numbers x, y is denoted by

$$E(r,s;x,y)=\begin{cases}\left(\dfrac{s}{r}\dfrac{x^r-y^r}{x^s-y^s}\right)^{\frac{1}{r-s}} & r\neq s, rs\neq 0, x\neq y,\\ L^{\frac{1}{r}}(x^r,y^r) & r\neq 0, s=0, x\neq y,\\ L^{\frac{1}{s}}(x^s,y^s) & r=0, s\neq 0, x\neq y,\\ E^{\frac{1}{r}}(x^r,y^r) & r=s\neq 0, x\neq y,\\ G(x,y) & r=s=0, x\neq y,\\ y & x=y,\end{cases} \tag{1.3}$$

where

$$L(x,y)=(x-y)\,/\ln(x/y), E(x,y)=e^{-1}(x^x/y^y)^{\frac{1}{x-y}}, G(x,y)=\sqrt{xy}.$$

In 1998, L. Losonczi and Zs. Páles showed that Minkowski type inequality hold for Stolarsky means (extended means) (see [3]).

Zhen-Hang Yang studied the AA, GG, AG and GA convexity of homogeneous functions of two variables, of which simplified decision methods were presented. As applications, new simple proofs of Hölder, Chebyshev and Minkowski type inequalities for homogeneous means such as $J(\alpha;x,y)$ and $S(\alpha;x,y)$ were given (see [7]).

The main purpose of this paper is to investigate the AA-convexity and GG-convexity of the Stolarsky means (extended means) by applying theorems or corollaries in [7], and then Hölder, Chebyshev and Minkowski type inequalities for Stolarsky means (extended means) are given.

Our main results are as follows:

Theorem 1 (Hölder Type Inequality for Stolarsky Mean). *For given $p,q>0$ with $p+q=1$ and arbitrary pairs $(x_1,y_1),(x_2,y_2)\in\mathbb{R}_+$, inequality*

$$E(r,s;x_1^px_2^q,y_1^py_2^q)\leq(\geq)E^p(r,s;x_1,y_1)E^q(r,s;x_2,y_2) \tag{1.4}$$

holds iff $r+s>(<)0.$ With equality iff $x_1:y_1=x_2:y_2$.

Theorem 2 (Chebyshev Type Inequality for Stolarsky Mean). *If (x_1,y_1) and (x_2,y_2) are oppositely (similarly) ordered, then*

$$E(r,s;x_1x_2,y_1y_2)\leq(\geq)E(r,s;x_1,y_1)E(r,s;x_2,y_2) \tag{1.5}$$

if $r+s>(<)0$. With equality iff $x_1=y_1$ or $x_2=y_2$.

Theorem 3 (Minkowski Type Inequality for Stolarsky Mean). *For arbitrary pairs $(x_1,y_1),(x_2,y_2)\in\mathbb{R}_+$, the following inequality*

$$E(r,s;x_1+x_2,y_1+y_2)\leq(\geq)E(r,s;x_1,y_1)+E(r,s;x_2,y_2) \tag{1.6}$$

holds iff $r+s\geq(\leq)3$ and $\min(r,s)\geq(\leq)1$ with $(r,s)\neq(2,1),(1,2)$. With equality iff $x_1:y_1=x_2:y_2$

To prove the above theorems, we need corollaries in [7] as preparations. For the sake of simplicity, we read them as follows:

Lemma 1. [7, Corollary 3]*Let $f:\mathbb{D}(\subseteq\mathbb{R}_+\times\mathbb{R}_+)\to\mathbb{R}_+$ is an n-order homogeneous and two-time differentiable function. Then for given $p,q>0$ with $p+q=1$ and arbitrary pairs $(x_1,y_1),(x_2,y_2),(x_1^px_2^q,y_1^py_2^q)\in\mathbb{D}$, inequality*

$$f(x_1^px_2^q,y_1^py_2^q)\leq f^p(x_1,y_1)f^q(x_2,y_2) \tag{1.7}$$

holds iff $I = (\ln f)_{xy} < 0$. *With equality iff* $x_1 : y_1 = x_2 : y_2$.

Lemma 2. [7, Corollay 5]*Let* $f : \mathbb{D}(\subseteqq \mathbb{R}_+ \times \mathbb{R}_+) \to \mathbb{R}_+$ *is an* n*-order homogeneous and two-time differentiable function. If* $I = (\ln f(x, y))_{xy} < (>)0$, *then for arbitrary pairs* $(x_1, y_1), (x_2, y_2)$ *and* $(1, 1) \in \mathbb{D}$ *we have*

$$f(1,1)f(x_1x_2, y_1y_2) < (>) f(x_1, y_1)f(x_2, y_2). \tag{1.8}$$

if (x_1, y_1) *and* (x_2, y_2) *are oppositely ordered. It is reversed if* (x_1, y_1) *and* (x_2, y_2) *are similarly ordered.*

Lemma 3. [7, Corollary 1]*Let* $f : \mathbb{D} \to \mathbb{R}$ *is a one-order homogeneous function and is two-time differentiable. Then for given* $pq > 0$ *with* $p+q = 1$ *and arbitrary pairs* $(x_1, y_1), (x_2, y_2), (px_1 + qx_2, py_1 + qy_2) \in \mathbb{D}$, *the following inequality*

$$f(px_1 + qx_2, py_1 + qy_2) \le pf(x_1, y_1) + qf(x_2, y_2) \tag{1.9}$$

holds iff $xyf_{xy} < 0$. *With equality iff* $x_1 : y_1 = x_2 : y_2$.

2. Proofs of Theorem 1 and 2

Proof of Theorem 1. By Lemma 1, to prove Theorem 1 and 2, it is enough to prove $I = \frac{\partial^2 \ln E}{\partial x \partial y} < (>)0$ for all $(x, y) \in \mathbb{R}_+ \times \mathbb{R}_+$. By partial derivative calculations, for $rs(r - s) \neq 0$ we get

$$\ln E = \frac{1}{r-s} \ln \left(\frac{s}{r} \frac{x^r - y^r}{x^s - y^s} \right), \tag{2.1}$$

$$\frac{\partial \ln E}{\partial x} = \frac{1}{E}\frac{\partial E}{\partial x} = \frac{1}{x(r-s)} \left(\frac{rx^r}{x^r - y^r} - \frac{sx^s}{x^s - y^s} \right), \tag{2.2}$$

$$\frac{\partial \ln E}{\partial y} = \frac{1}{E}\frac{\partial E}{\partial y} = \frac{1}{y(r-s)} \left(\frac{-ry^r}{x^r - y^r} + \frac{sy^s}{x^s - y^s} \right), \tag{2.3}$$

$$\frac{\partial^2 \ln E}{\partial x \partial y} = \frac{1}{xy(r-s)} \left(\frac{r^2x^ry^r}{(x^r - y^r)^2} - \frac{s^2x^sy^s}{(x^s - y^s)^2} \right). \tag{2.4}$$

1) For $rs(r - s) \neq 0$. Put $x/y = u, \ln u = t, t \in \mathbb{R}$ and note $\sinh x = \frac{1}{2}(e^x - e^{-x}), \cosh x = \frac{1}{2}(e^x + e^{-x})$, then I can be denoted by

$$\begin{aligned} I &= \frac{1}{xy(r-s)} \left(\frac{r^2}{4\sinh^2 \frac{r}{2}t} - \frac{s^2}{4\sinh^2 \frac{s}{2}t} \right) \\ &= \frac{r^2s^2(xy)^{-1}(r-s)^{-1}}{4\sinh^2 \frac{r}{2}t \sinh^2 \frac{s}{2}t} \left(\frac{\sinh^2 \frac{s}{2}t}{s^2} - \frac{\sinh^2 \frac{r}{2}t}{r^2} \right) \\ &= c_1 g(t), \end{aligned}$$

where $c_1 = \frac{r^2s^2(xy)^{-1}}{4\sinh^2 \frac{r}{2}t \sinh^2 \frac{s}{2}t}, g(t) = \frac{1}{r-s}\left(\frac{\sinh^2 \frac{s}{2}t}{s^2} - \frac{\sinh^2 \frac{r}{2}t}{r^2} \right)$. Obviously, $\operatorname{sgn} c_1 = 1$, which leads to $\operatorname{sgn} I = \operatorname{sgn} c_1 \operatorname{sgn} g(t) = \operatorname{sgn} g(t)$.

Note

$$\begin{aligned} g'(t) &= \frac{1}{2(r-s)} \left(\frac{\sinh st}{s} - \frac{\sinh rt}{r} \right), \\ g''(t) &= -\frac{\cosh st - \cosh rt}{s-r}. \end{aligned}$$

From Taylor formula, there exist ξ lies between 0 and t such that

$$g(t)=g(0)+g'(0)t+\frac{1}{2}g''(\xi)t^2=\frac{1}{2}g''(\xi)t^2,$$

which implies

$$\begin{aligned}\operatorname{sgn} I &= \operatorname{sgn} g(t)=\operatorname{sgn} g''(t)\\ &= -\operatorname{sgn}\frac{\cosh st-\cosh rt}{s-r}=-\operatorname{sgn}(s+r),\end{aligned}$$

i.e.

$$\operatorname{sgn} I=\begin{cases}-1, & s+r>0;\\ 0, & s+r=0;\\ 1, & s+r<0.\end{cases}$$

Thus we derive our required results.

2) For $rs(r-s)=0$. For instance $r=0, s\neq 0$, define that

$$I(0,s;t):=\lim_{r\to 0} I(r,s;t).$$

For r close to 0 by the above proof, we have $\operatorname{sgn}(c_1)=1$ and $\operatorname{sgn} g''(t)=-\operatorname{sgn}(s+r)$, and then $I(r,s;t)<(>)0$ iff $s+r>(<)0$, hence $I(0,s;t)<(>)0$ iff $s+0>(<)0$.

Likewise, we still have that $I<(>)0$ for all $(x,y)\in\mathbb{R}_+\times\mathbb{R}_+$ iff $r+s>(<)0$ in the case of $s=0, r\neq 0$ or $r=s\neq 0$, Of which the processes are omitted.

This proof is completed. ∎

Proof of Theorem 2. By Lemma 2 and Theorem 1, note $E(r,s;1,1)=1$, we immediately obtain the required results.

This proof is completed. ∎

3. Proof of Theorem 3

Since $E(r,s;x,y)$ is a one-order homogeneous function of variables x and y, to prove Theorem 3, it is enough to prove $E_{xy}=\frac{\partial^2 E(r,s;x,y)}{\partial x\partial y}<(>)0$ for all $(x,y)\in\mathbb{R}_+\times\mathbb{R}_+$.

By (2.2)-(2.4) and Note $\frac{\partial^2 \ln E}{\partial x\partial y}=\frac{1}{E^2}\left(E\frac{\partial^2 E}{\partial x\partial y}-\frac{\partial E}{\partial x}\frac{\partial E}{\partial y}\right)$, for $rs(r-s)\neq 0$ we get

$$\begin{aligned}\frac{1}{E}\frac{\partial^2 E}{\partial x\partial y} &= \frac{\partial^2 \ln E}{\partial x\partial y}+(\frac{1}{E}\frac{\partial E}{\partial x})(\frac{1}{E}\frac{\partial E}{\partial y})\\ &= \frac{1}{xy(r-s)}\left[\tfrac{r^2x^ry^r}{(x^r-y^r)^2}-\tfrac{s^2x^sy^s}{(x^s-y^s)^2}\right]\\ &\quad+\frac{1}{xy(r-s)^2}\left[\tfrac{-r^2x^ry^r}{(x^r-y^r)^2}+\tfrac{rs(x^ry^s+x^sy^r)}{(x^r-y^r)(x^s-y^s)}+\tfrac{-s^2x^sy^s}{(x^s-y^s)^2}\right]\\ &= \frac{1}{xy(r-s)^2}\left[\tfrac{r^2(r-s-1)x^ry^r}{(x^r-y^r)^2}+\tfrac{rs(xy)^s(x^{r-s}+y^{r-s})}{(x^r-y^r)(x^s-y^s)}-\tfrac{s^2(r-s+1)x^sy^s}{(x^s-y^s)^2}\right];\end{aligned} \tag{3.1}$$

put $x/y = u, \ln u = t, t \in \mathbb{R}$, and note $\sinh x = \frac{1}{2}(e^x - e^{-x}), \cosh x = \frac{1}{2}(e^x + e^{-x})$, then $\frac{1}{E}\frac{\partial^2 E}{\partial x \partial y}$ can be expressed as

$$\begin{aligned} \frac{1}{E}\frac{\partial^2 E}{\partial x \partial y} &= \frac{1}{xy(r-s)^2}\left[\frac{r^2(r-s-1)}{4\sinh^2\frac{r}{2}t} + \frac{2rs\cosh\frac{r-s}{2}t}{4\sinh\frac{r}{2}t\sinh\frac{s}{2}t} - \frac{s^2(r-s+1)}{4\sinh^2\frac{s}{2}t}\right] \\ &= \frac{\sinh^{-2}\frac{r}{2}t\sinh^{-2}\frac{s}{2}t}{4xy(r-s)^2}[r^2(r-s-1)\sinh^2\tfrac{s}{2}t \\ &\quad +2rs\cosh\tfrac{r-s}{2}t\sinh\tfrac{r}{2}t\sinh\tfrac{s}{2}t - s^2(r-s+1)\sinh^2\tfrac{r}{2}t], \end{aligned}$$

using the following well-known equations:

$$\begin{cases} \sinh x \sinh y = \frac{1}{2}\left[\cosh(x+y) - \cosh(x-y)\right], \\ \cosh x \cosh y = \frac{1}{2}\left[\cosh(x+y) + \cosh(x-y)\right], \\ \sinh^2 x = \frac{1}{2}(\cosh 2x - 1), \\ \cosh^2 x = \frac{1}{2}(\cosh 2x + 1), \end{cases}$$

then $\frac{1}{E}\frac{\partial^2 E}{\partial x \partial y}$ can be concisely expressed as

$$\frac{1}{E}\frac{\partial^2 E}{\partial x \partial y} = c_0[A\cosh rt + B\cosh st + C\cosh(r-s)t + D], \tag{3.2}$$

where

$$\begin{cases} c_0 = -\frac{1}{8xy}(r-s)^{-2}\sinh^{-2}\frac{r}{2}t\sinh^{-2}\frac{s}{2}t, \\ A = s(s-1)(r-s), \\ B = -r(r-1)(r-s), \\ C = rs, \\ D = (r-s)^2(r+s) - (r^2 - rs + s^2). \end{cases}$$

Next let us prove stepwise.

Lemma 4 (Step 1)**.** *Let*

$$g(t) = A\cosh rt + B\cosh st + C\cosh(r-s)t + D. \tag{3.3}$$

Then $\frac{\partial^2 E}{\partial x \partial y} < (>)0$ *for all* $(x, y) \in \mathbb{R}_+ \times \mathbb{R}_+$ *iff* $g''(t) > (<)0$.

Proof. By simple calculations we get $g(0) = g'(0) = 0$. Using Taylor formula, for arbitrary $t \in \mathbb{R}$ we have

$$g(t) = g''(\xi)\frac{t^2}{2}, \xi \text{ lie in } 0 \text{ and } t.$$

This shows $g(t) > (<)0$ for all $t \in \mathbb{R}$ iff $g''(t) > (<)0$ for all $t \in \mathbb{R}$. From $E, c_0 > 0$ and $\frac{1}{E}\frac{\partial^2 E}{\partial x \partial y} = c_0 g(t)$ it follows that $\frac{\partial^2 E}{\partial x \partial y} < (>)0$ for all $(x, y) \in \mathbb{R}_+ \times \mathbb{R}_+$ iff $g''(t) > (<)0$.

The Lemma is proved. ∎

Lemma 5 (Step 2)**.** *If* $(r+s)(r-s) \neq 0$, *then* $g''(t)$ *can be denoted by*

$$g''(t) = rs(r-s)p_1(t)p_2(t), \tag{3.4}$$

where

$$p_1(t) = \cosh rt - \cosh st, \tag{3.5}$$

$$p_2(t) = \begin{cases} r(s-1) + (r-s)\frac{\cosh(r-s)t - \cosh st}{\cosh rt - \cosh st}, & t \neq 0; \\ \frac{rs}{r+s}(r+s-3), & t = 0. \end{cases} \tag{3.6}$$

Proof. Note

$$Ar^2 + Bs^2 + C(r-s)^2 = 0, \tag{3.7}$$

by simple derivative calculations, for $t \neq 0$ we get

$$\begin{aligned} g''(t) &= Ar^2 \cosh rt + Bs^2 \cosh st + C(r-s)^2 \cosh(r-s)t \\ &= Ar^2(\cosh rt - \cosh st) + C(r-s)^2[\cosh(r-s)t - \cosh st] \quad (3.8) \\ &= (\cosh rt - \cosh st)\left[Ar^2 + C(r-s)^2 \tfrac{\cosh(r-s)t - \cosh st}{\cosh rt - \cosh st}\right] \\ &= rs(r-s)(\cosh rt - \cosh st)\left[r(s-1) + (r-s)\tfrac{\cosh(r-s)t - \cosh st}{\cosh rt - \cosh st}\right] \\ &= rs(r-s)p_1(t)p_2(t); \\ g''(0) &= Ar^2 + Bs^2 + C(r-s)^2 = 0 = rs(r-s)p_1(0)p_2(0). \end{aligned}$$

This completes the proof. ∎

Lemma 6 (Step 3)**.** *$p_2(t) \geq (\leq)0$ for all $t \in \mathbb{R}$ iff $p_2(0) \geq (\leq)0$ and $p_2(\infty) \geq (\leq)0$.*

Proof. 1) we first consider the monotone of $p_2(t)$. By derivative calculations, we get

$$p_2'(t) = (r-s)\tfrac{\cosh(r-s)t - \cosh st}{\cosh rt - \cosh st}\left[\tfrac{(r-s)\sinh(r-s)t - s\sinh st}{\cosh(r-s)t - \cosh st} - \tfrac{r\sinh rt - s\sinh st}{\cosh rt - \cosh st}\right]. \tag{3.9}$$

Since $x\sinh x = |x|\sinh|x|, \cosh x = \cosh|x|$, so $p_2'(t)$ can be expressed as

$$p_2'(t) = \frac{r-s}{|t|}\tfrac{\cosh(r-s)t - \cosh st}{\cosh rt - \cosh st}\left[\tfrac{|(r-s)t|\sinh|(r-s)t| - |st|\sinh|st|}{\cosh|(r-s)t| - \cosh|st|} - \tfrac{|rt|\sinh|rt| - |st|\sinh|st|}{\cosh|rt| - \cosh|st|}\right]. \tag{3.10}$$

It is easy to prove that for $t_i > 0, i = 1, 2, 3$ and are pairwise unequal, then

$$\operatorname{sgn}\left(\tfrac{t_1\sinh t_1 - t_2\sinh t_2}{\cosh t_1 - \cosh t_2} - \tfrac{t_2\sinh t_2 - t_3\sinh t_3}{\cosh t_2 - \cosh t_3}\right) = \operatorname{sgn}(\cosh t_1 - \cosh t_3). \tag{3.11}$$

In fact, for $f(x) = \sqrt{x^2-1}\ln(x+\sqrt{x^2-1})$ we have $f''(x) > 0$ on $(1, +\infty)$. By simple derivative calculations, we get

$$\begin{aligned} f'(x) &= 1 + \frac{x}{\sqrt{x^2-1}}\ln(x+\sqrt{x^2-1}), \\ f''(x) &= \frac{x\sqrt{x^2-1} - \ln(x+\sqrt{x^2-1})}{(\sqrt{x^2-1})^3} = \frac{h(x)}{(\sqrt{x^2-1})^3} \\ h'(x) &= \frac{2}{\sqrt{x^2-1}} > 0, \end{aligned}$$

and then $h(x) > h(1) = 0$. It follows that $f''(x) > 0$. According to the property of convex functions, there are

$$\frac{1}{x_1 - x_3}\left[\frac{f(x_1) - f(x_2)}{x_1 - x_2} - \frac{f(x_2) - f(x_3)}{x_2 - x_3}\right] > 0, \tag{3.12}$$

i.e.

$$\operatorname{sgn}\left[\frac{f(x_1) - f(x_2)}{x_1 - x_2} - \frac{f(x_2) - f(x_3)}{x_2 - x_3}\right] = \operatorname{sgn}(x_1 - x_3), \tag{3.13}$$

where x_1, x_2 and x_3 are pairwise unequal.

Put $x_i = \cosh t_i$ with $t_i > 0, i = 1, 2, 3$, then $\sqrt{x_i^2 - 1} = \sinh t_i, \ln(x_i + \sqrt{x_i^2 - 1}) = t_i$. Consequently (3.13) can be changed into (3.11), by which and (3.10) we get

$$\begin{aligned} \operatorname{sgn} p_2'(t) &= \operatorname{sgn}\left(\frac{r-s}{|t|}\right) \operatorname{sgn}\left[\tfrac{\cosh(r-s)t - \cosh st}{\cosh rt - \cosh st}\right] \operatorname{sgn}[\cosh|(r-s)t| - \cosh|rt|] \\ &= \operatorname{sgn}\left(\frac{r-s}{|t|}\right) \operatorname{sgn}\left[\tfrac{r(r-2s)}{(r-s)(r+s)}\right] \operatorname{sgn}[-s(2r-s)] \\ &= -\operatorname{sgn}(rs)\operatorname{sgn}(r+s)\operatorname{sgn}(r-2s)\operatorname{sgn}(2r-s). \qquad (3.14) \end{aligned}$$

This shows $p_2(t)$ is always monotone in $t \in \mathbb{R}$.

2) Next let us observe that $p_2(0)$ and $p_2(\infty)$. It is easy to verify

$$p_2(0) \ := \lim_{t\to+0} p_2(t) = \frac{rs}{r+s}(r+s-3), \qquad (3.15)$$

$$p_2(\infty) \ := \lim_{t\to\infty} p_2(t) = \begin{cases} r(s-1), & r > s > 0; \\ \infty, & r > 0 > s, r+s > 0; \\ -\infty, & r > 0 > s, r+s < 0; \\ s(r-1), & 0 > r > s. \end{cases} \qquad (3.16)$$

3) It is easy to verify that $p_2(t)$ is an even function on $\mathbb{R}$. And then because $p_2(t)$ is always monotone in $t \in \mathbb{R}$ we obtain immediately that $p_2(t) \geq 0$ for all $t \in \mathbb{R}$ iff $p_2(0) \geq 0$ and $p_2(\infty) \geq 0$, and $p_2(t) \leq 0$ for all $t \in \mathbb{R}$ iff $p_2(0) \leq 0$ and $p_2(\infty) \leq 0$. Thus we complete step 3. ∎

Lemma 7 (Step 4)**.** *For $rs(r-s) \neq 0$ we have*

1) $g''(t) \geq 0$ for all $t \in \mathbb{R}$ iff

$$r + s \geq 3 \text{ and } \min(r, s) \geq 1. \qquad (3.17)$$

2) $g''(t) \leq 0$ for all $t \in \mathbb{R}$ iff

$$r + s \leq 3 \text{ and } \min(r, s) \leq 1. \qquad (3.18)$$

Proof. Without loss of generality, we assume $r > s$. Note

$$\begin{aligned} \operatorname{sgn}(p_1(t)) &= \operatorname{sgn}(\cosh rt - \cosh st) \\ &= \operatorname{sgn}(rt + st)\operatorname{sgn}(rt - st) \\ &= \operatorname{sgn}(r-s)\operatorname{sgn}(r+s). \qquad (3.19) \end{aligned}$$

Next let us consider three cases:

1) For $r + s > 0$. By our assumption and (3.19) there is $p_1(t) > 0$. From (3.4) and Step 3, we see that $g''(t) \geq 0, t \in \mathbb{R}$ iff $rsp_2(t) \geq 0$ for all $t \in \mathbb{R}$ iff $rsp_2(0) \geq 0$ and $rsp_2(\infty) \geq 0$. It follows from (3.15) and (3.16) that $g''(t) \geq 0, t \in \mathbb{R}$ iff

$$r + s \geq 3 \text{ and } r > s \geq 1.$$

Likewise $g''(t) \leq 0, t \in \mathbb{R}$ iff

$$r + s \leq 3 \text{ and } r > s > 0, s \leq 1 \text{ or } r > 0 > s. \qquad (3.20)$$

2) For $r + s < 0$. By our assumption and (3.19) there is $p_1(t) < 0$. From (3.4) and Step 3, we see that $g''(t) \geq 0, t \in \mathbb{R}$ iff $rsp_2(t) \leq 0$ for all $t \in \mathbb{R}$ iff $rsp_2(0) \leq 0$ and $rsp_2(\infty) \leq 0$. From (3.15) and (3.16) we see that is impossible.

That $g''(t) \le 0, t \in \mathbb{R}$ iff $rsp_2(t) \ge 0$ for all $t \in \mathbb{R}$ iff $rsp_2(0) \ge 0$ and $rsp_2(\infty) \ge 0$. It follows from (3.15) and (3.16) that

$$r > 0 > s \text{ or } 0 > r > s. \tag{3.21}$$

3) For $r + s = 0$. By (3.8) we have,

$$\begin{aligned} g''(t) &= C(r-s)^2[\cosh(r-s)t - \cosh st] \\ &= -s^2(r-s)^2[\cosh(-2st) - \cosh(st)], \end{aligned} \tag{3.22}$$

and then

$$\operatorname{sgn}(g''(t)) = -\operatorname{sgn}(s^2(r-s)^2)\operatorname{sgn}(3s)\operatorname{sgn}(s) = -1.$$

Combining 1), 2) with 3), we immediately get the desired results.

Step 4 is completed. ∎

Lemma 8 (Step 5)**.** *For $rs(r-s) \ne 0$ we have $g''(t) = 0$ for all $t \in \mathbb{R}$ iff $(r,s) = (2,1), (1,2)$.*

Proof. We still assume $r > s$. It is obvious that $g''(t)$ is not always zero for all $t \in \mathbb{R}$ if $r + s = 0$ by (3.22) and need only consider the case of $r + s \ne 0$.

If $(r,s) = (2,1)$, then by a direct calculation there is $g''(t) = 0$ for all $t \in \mathbb{R}$.

If $g''(t) = 0$ for all $t \in \mathbb{R}$, then there must hold $p_2(t) = 0$ for all $t \in \mathbb{R}$ since $g''(t) = rs(r-s)p_1(t)p_2(t)$, where $p_2(0)$ is defined by (3.15). By (3.14) there must hold $r = 2s$ or $2r = s$.

On the other hand, by $p_2(t) = p_2(0) = p_2(+\infty) = 0$ we can find $s = 1, r = 2$.

Step 5 is completed. ∎

Lemma 9 (Step 6)**.** *For $rs(r-s) = 0$, we still have $\frac{\partial^2 E(r,s;x,y)}{\partial x \partial y} \le (\ge) 0$ for all $(x,y) \in \mathbb{R}_+ \times \mathbb{R}_+$ iff*

$$r + s \ge (\le) 3 \textit{ and } \min(r,s) \ge (\le) 1.$$

Proof. If $r = 0, s(r-s) \ne 0$, then

$$\begin{aligned} E_{xy}(0,s;x,y) &= \frac{\partial^2[\lim\limits_{r\to 0} E(r,s;x,y)]}{\partial x \partial y} = \lim_{r\to 0} \frac{\partial^2 E(r,s;x,y)}{\partial x \partial y} \\ &= \lim_{r\to 0} E \frac{\partial^2 E(r,s;x,y)}{\partial x \partial y} = \lim_{r\to 0} \left(E \cdot c_0 g(t)\right). \end{aligned}$$

For r close to 0 $\operatorname{sgn}(c_0) = 1$, by Step 1 and 4 we have $\frac{\partial^2 E(r,s;x,y)}{\partial x \partial y} \le (\ge) 0$ iff $g(t) \ge (\le) 0$ iff $g''(t) \ge (\le) 0$ iff both $r + s \ge (\le) 3$ and $\min(r,s) \ge (\le) 1$ hold. This shows that $\frac{\partial^2 E(0,s;x,y)}{\partial x \partial y} \le (\ge) 0$ iff $0 + s \ge (\le) 3$ and $\min(0,s) \ge (\le) 1$, i.e. this Lemma is valid.

Likewise, this Lemma is also true in the cases of $s = 0, r(r-s) \ne 0$ and $rs \ne 0, r - s = 0$ and $r = s = 0$.

This proof is completed. ∎

Proof of Theorem 3. For $rs(r-s) \ne 0$ with $(r,s) \ne (2,1), (1,2)$, by Step1, 4 and 5, we have $E_{xy} = \frac{\partial^2 E(r,s;x,y)}{\partial x \partial y} < (>) 0$ iff $r + s \ge (\le) 3$ and $\min(r,s) \ge (\le) 1$; which is also valid for $rs(r-s) = 0$ by Step 6. Note $x, y > 0$, applying Lemma 3, we immediately obtain the required results.

This completes the proof. ∎

Remark 1. *In fact, the above proof is another new proof of Minkowski's inequality for Stolarsky means except what L. Losonczi and Zs. Páles have given.*

References

[1] H. Alzer, Üer eine einparametrige familie von Mitlewerten, *Bayer. Akad. Wiss. Math.–Natur. Kl. Sitzungsber.* **1987**(1988), 23–29. (German)

[2] H. Alzer, Üer eine einparametrige familie von Mitlewerten, II, *Bayer. Akad. Wiss. Math.–Natur. Kl. Sitzungsber.* **1988**(1989), 23–29. (German)

[3] L. Losonczi AND Zs. Páles, Minkowski's inequality for two variable difference means, *Proc. Amer. Math. Soc.* **126**(1998), 779-789.

[4] Hongwei Lou, Hlder inequalities of means, *J. of Ningbo Univ.*, **9**(1)(1996), 1-7.

[5] J. Sándor, On Lehman's inequality and electrical networks, *RGMIA Research Report Collection*, **8**(1), Article 1, 2005.

[6] Zhen-Hang Yang, Holder,Minkowski, and Chebyshev Type Inequalities for Logarithmic Mean and Exponential Mean, *J. of Xuzhou Normal Univ. (Natural Science Edition)*, **23**(1)(2005), 31-34.

[7] Zhen-Hang Yang, Minkowski, Hölder and Chebyshev Type Inequality of Homogeneous Functions (manuscript).

Zhejiang Electric Power Vocational Technical College, Hangzhou, Zhejiang, China, 311600

E-mail address: yzhkm@163.com