An Improvement of the Harmonic-Geometric Inequality

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Let $x = (x_1, ..., x_n)$, where $x_i > 0$, and put $A = A(x) = \frac{x_1 + ... + x_n}{n}$, $G = G(x) = \sqrt{x_1 ... x_n}$, $H = H(x) = \frac{n}{\frac{1}{x_1} + ... + \frac{1}{x_n}}$ for the arithmetic, geometric, and harmonic means of x_i $\left(i = \overline{1, n}\right)$. Put

(1). $M(x) = \frac{G(x) \sqrt[n]{A(x)}}{A\left(\sqrt[n]{x}\right)}$

where $\sqrt[n]{x} = \left(\sqrt[n]{x_1}, ..., \sqrt[n]{x_n}\right)$. Since $G\left(\frac{1}{x}\right) = \frac{1}{G(x)}$, $A\left(\frac{1}{x}\right) = \frac{1}{H(x)}$, for $\frac{1}{x} = \left(\frac{1}{x_1}, ... \frac{1}{x_n}\right)$, clearly the inequality

(2). $G \leq M(x) \leq A$

is equivalent to

(3). $H \leq N(x) \leq G$,

where

(4). $N(x) = \frac{G(x) \sqrt[n]{H(x)}}{H\left(\sqrt[n]{x}\right)}$

We will prove that (3) holds true (i.e. (2), too), even with a chain of improvements.

Theorem. One has the inequalities

(5). $H \leq N_2(x) \leq N_1(x) \leq N(x) \leq G$

where $N_1(x) = \frac{H \sqrt[n]{A}}{H\left(\sqrt[n]{x}\right)}$, $N_2(x) = \frac{H \cdot A\left(\sqrt[n]{x}\right)}{H\left(\sqrt[n]{x}\right)}$, with $H = H(x)$, etc and $N(x)$ given by (4).

Proof. We will apply the famous Sierpinski inequality (see [1], with a generalization), which can be written as follows:

(6). $H^{n-1} A \leq G^n \leq A^{n-1} H$

From the left side of (6) we get $\frac{H^n A}{H} \leq G^n$, i.e. $H \sqrt[n]{A} \leq G \sqrt[n]{H}$, thus

(*). $\frac{G \sqrt[n]{H}}{H} \geq \sqrt[n]{A}$

Now, the inequality $\sqrt[n]{A(x)} \geq A\left(\sqrt[n]{x}\right)$ is valid, being equivalent to $\sqrt[n]{\frac{1}{n} \sum\limits_{k=1}^{n} x_k} \geq \frac{1}{n} \sum\limits_{k=1}^{n} \sqrt[n]{x_k}$, and this follows by Jensen‘s classical inequality, for the concave function $f(x) = \sqrt[n]{x}$ $(x > 0, n \geq 1)$. Now $A\left(\sqrt[n]{x}\right) \geq H\left(\sqrt[n]{x}\right)$, so by (*), we get the relation

(**). $\frac{G \sqrt[n]{H}}{H} \geq \sqrt[n]{A} \geq A\left(\sqrt[n]{x}\right) \geq H\left(\sqrt[n]{x}\right)$

Thus, we get $\frac{G \sqrt[n]{H}}{H\left(\sqrt[n]{x}\right)} \geq H$, but by (**), even two refinements of this inequalities, can be deduced. The right-side of (2) is $\sqrt[n]{H(x)} \leq H\left(\sqrt[n]{x}\right)$, which with $x \to \frac{1}{x}$ becomes in fact $\sqrt[n]{A(x)} \geq A\left(\sqrt[n]{x}\right)$, and this is true, as we have pointed out before.

Thus the theorem follows.

We note that many similar refinements as in (5) are true, but they will be examined in a further, using paper.

References.

[1]. J. Sándor, On inequality of Sierpinski, Octogon Mathematical Magazine, Vol. 3, No. 1, April 1995, pp. 21-22.

[2]. Collection Octogon Mathematical Magazine (1993-2005).