ON IMPROVEMENTS OF THE POWER MEAN INEQUALITY

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Abstract

In this paper, we give some improvements of power mean inequality $M_r(a_i : 1 \leq i \leq n) \leq M_p(a_i : 1 \leq i \leq n)$, where $0 < r \leq p$, $M_p(a_i : 1 \leq i \leq n) = (\frac{1}{n}\sum_{i=1}^{n} a_i^p)^{\frac{1}{p}}$. These inequalities can be applied in estimation of convexity modulus.

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Let $a_i (i = 1, 2, \cdots, n)$ be nonnegative real numbers, then the power mean is given by

$$M_p(a_i : 1 \leq i \leq n) = \left(\frac{1}{n}\sum_{i=1}^{n} a_i^p\right)^{\frac{1}{p}}.$$

For power mean, we have the well-known power mean inequality (See [1]):

$$M_r(a_i : 1 \leq i \leq n) \leq M_p(a_i : 1 \leq i \leq n), \quad 0 < r \leq p.$$

In this paper, we give some improvements of the power mean inequality.

Theorem 1 *Let $a_i (1 \leq i \leq n)$ be nonnegative real numbers, $0 < r \leq 1$ and $p \geq 2$. Then*

$$M_r^{pr}(a_i : 1 \leq i \leq n) \leq M_p^{pr}(a_i : 1 \leq i \leq n) - \frac{n^{2-p}}{2} M_p^p(|a_i^r - a_j^r| : 1 \leq i, j \leq n),$$

that is

$$\left(\frac{1}{n}\sum_{i=1}^{n} a_i^r\right)^p \leq \left(\frac{1}{n}\sum_{i=1}^{n} a_i^p\right)^r - \frac{1}{2n^p}\sum_{i=1}^{n}\sum_{j=1}^{n} |a_i^r - a_j^r|^p. \tag{1}$$

Proof Let $a_i^r = b_i$, then (1) can be translated into

$$\left(\sum_{i=1}^{n} b_i\right)^p \le n^{p-r}\left(\sum_{i=1}^{n} b_i^{\frac{p}{r}}\right)^r - \sum_{1\le i<j\le n} |b_i - b_j|^p.$$

Since $b_1, b_2, \cdots, b_n$ are symmetric in the above inequality, without losing the generality, we suppose $b_1 \ge b_2 \ge \cdots \ge b_n \ge 0$. Let

$$\begin{aligned} f_1(t) &= \left(t + \sum_{i=2}^{n} b_i\right)^p - n^{p-r}\left(t^{\frac{p}{r}} + \sum_{i=2}^{n} b_i^{\frac{p}{r}}\right)^r \\ &\quad + \sum_{i=2}^{n}(t-b_i)^p + \sum_{2\le i<j\le n}(b_i - b_j)^p, \quad t \ge b_2, \end{aligned}$$

by derivation, we obtain

$$f_1'(t) = p\left(t + \sum_{i=2}^{n} b_i\right)^p - pn^{p-r}t^{\frac{p}{r}-1}\left(t^{\frac{p}{r}} + \sum_{i=2}^{n} b_i^{\frac{p}{r}}\right)^{r-1} + p\sum_{i=2}^{n}(t-b_i)^{p-1}.$$

Setting

$$g(x_2, \cdots, x_n) = p\left(t + \sum_{i=2}^{n} x_i\right)^{p-1} - pn^{p-r}t^{\frac{p}{r}-1}\left(t^{\frac{p}{r}} + \sum_{i=2}^{n} x_i^{\frac{p}{r}}\right)^{r-1} + p\sum_{i=2}^{n}(t-x_i)^{p-1},$$

where $t \ge x_k \ge 0 (k = 2, 3, \cdots, n)$. By calculating the partial derivative, we have

$$\begin{aligned} g'_{x_k}(x_2, \cdots, x_n) &= p(p-1)\left(t + \sum_{i=2}^{n} x_i\right)^{p-2} \\ &\quad - p(r-1)\frac{p}{r}t^{\frac{p}{r}-1}x_k^{\frac{p}{r}-1}\left(t^{\frac{p}{r}} + \sum_{i=2}^{n} x_i^{\frac{p}{r}}\right)^{r-2} - p(p-1)(t-x_i)^{p-2} \\ &= p(p-1)\left[\left(t + \sum_{i=2}^{n} x_i\right)^{p-2} - (t-x_i)^{p-2}\right] \\ &\quad + p^2\frac{1-r}{r}t^{\frac{p}{r}-1}x_k^{\frac{p}{r}-1}\left(t^{\frac{p}{r}} + \sum_{i=2}^{n} x_i^{\frac{p}{r}}\right)^{r-2} \\ &\ge 0. \end{aligned}$$

Thus, $g(x_2, \cdots, x_n)$ is a increasing function for $x_k (k = 2, \cdots, n)$ on $[0, t]$. Hence, we have

$$\begin{aligned} f_1'(t) &\le g(b_2, \cdots, b_n) \le g(t, \cdots, t) = pn^{p-1}t^{p-1} - pn^{p-1}t^{\frac{p}{r}-1}n^{r-1}t^{\frac{p}{r}(r-1)} \\ &= pn^{p-1}t^{p-1} - pn^{p-1}t^{p-1} = 0. \end{aligned}$$

Thus, $f_1(t)$ is a decreasing function on $[a_2, +\infty]$. Hence, we have

$$f_1(b_1) \le f_1(b_2),$$

where

$$\begin{aligned} f_1(b_2) &= \left(2b_2 + \sum_{i=3}^{n} b_i\right)^p - n^{p-r}\left(2b_2^{\frac{p}{r}} + \sum_{i=3}^{n} b_i^{\frac{p}{r}}\right)^r \\ &\qquad + 2\sum_{i=3}^{n}(b_2 - b_i)^p + \sum_{3\le i<j\le n}(b_i - b_j)^p, \end{aligned}$$

setting

$$\begin{aligned} f_2(t) &= \left(2t + \sum_{i=3}^{n} b_i\right)^p - n^{p-r}\left(2t^{\frac{p}{r}} + \sum_{i=3}^{n} b_i^{\frac{p}{r}}\right)^r \\ &\qquad + 2\sum_{i=3}^{n}(t - b_i)^p + \sum_{3\le i<j\le n}(b_i - b_j)^p, \end{aligned}$$

by calculating the derivative, we obtain

$$\begin{aligned} f_2'(t) &= 2p\left(2t + \sum_{i=3}^{n} b_i\right)^{p-1} - 2pn^{p-r}t^{\frac{p}{r}-1}\left(2t^{\frac{p}{r}} + \sum_{i=3}^{n} b_i^{\frac{p}{r}}\right)^{r-1} + 2p\sum_{i=3}^{n}(t - b_i)^p \\ &\le 2pn^{p-1}t^{p-1} - 2pn^{p-r}t^{\frac{p}{r}-1}n^{r-1}t^{\frac{p}{r}(r-1)} = 0. \end{aligned}$$

Thus, $f_2(t)$ is a decreasing function on $[b_3, +\infty]$. Hence, we have

$$f_2(b_2) \le f_2(b_3),$$

however $f_1(b_2) = f_2(b_2)$, then

$$f_1(b_1) \le f_2(b_3).$$

By this method, we can obtain finally

$$f_1(b_1) \le f_{n-1}(b_n),$$

$$f_{n-1}(t) = [(n-1)t + b_n]^p - n^{p-r}\left[(n-1)t^{\frac{p}{r}} + b_n^{\frac{p}{r}}\right]^r + (n-1)(t - b_n)^p.$$

Hence, $f_1(b_1) < 0$. This is the inequality (1). The theorem is proved.

Theorem 2 *Let $a_i (1 \le i \le n)$ be nonnegative real numbers, and $1 \le r \le p \le 2$. Then*

$$M_r^{pr}(a_i : 1 \le i \le n) \le n^{2-p} M_p^{pr}(a_i : 1 \le i \le n) - \frac{n^{2-p}}{2} M_p^p(|a_i^r - a_j^r| : 1 \le i, j \le n),$$

that is

$$\left(\frac{1}{n}\sum_{i=1}^{n} a_i^r\right)^p \le n^{2-p}\left(\frac{1}{n}\sum_{i=1}^{n} a_i^p\right)^r - \frac{1}{2n^p}\sum_{i=1}^{n}\sum_{j=1}^{n}|a_i^r - a_j^r|^p. \tag{2}$$

Proof Let $a_i^r = b_i (i = 1, 2, \cdots, n)$, and rewrite (2) as

$$\left(\sum_{i=1}^n b_i\right)^p \le n\left(\sum_{i=1}^n b_i^{\frac{p}{r}}\right)^r - \sum_{1\le i<j\le n} |b_i - b_j|^p.$$

Without losing the generality, we suppose $b_1 \ge b_2 \ge \cdots \ge b_n \ge 0$. Let

$$\begin{aligned} F_1(t) &= \left(t + \sum_{i=2}^n b_i\right)^p - n\left(t^{\frac{p}{r}} + \sum_{i=2}^n b_i^{\frac{p}{r}}\right)^r + \sum_{i=2}^n (t - b_i)^p \\ &\quad + \sum_{2\le i<j\le n} (b_i - b_j)^p, \quad t \ge b_2 \end{aligned}$$

by derivation, we obtain

$$F_1'(t) = p\left(t + \sum_{i=2}^n b_i\right)^{p-1} - pnt^{\frac{p}{r}-1}\left(t^{\frac{p}{r}} + \sum_{i=2}^n b_i^{\frac{p}{r}}\right)^{r-1} + p\sum_{i=2}^n (t - b_i)^{p-1}.$$

Denote

$$\begin{aligned} G(x_2, \cdots, x_n) &= p\left(t + \sum_{i=2}^n x_i\right)^{p-1} - pnt^{\frac{p}{r}-1}\left(t^{\frac{p}{r}} + \sum_{i=2}^n x_i^{\frac{p}{r}}\right)^{r-1} \\ &\quad + p\sum_{i=2}^n (t - x_i)^{p-1}, \end{aligned}$$

where $t \ge x_k \ge 0 (k = 2, 3, \cdots, n)$.

$$\begin{aligned} G'_{x_k}(x_2, \cdots, x_n) &= p(p-1)\left(t + \sum_{i=2}^n x_i\right)^{p-2} \\ &\quad - pn\frac{p}{r}(r-1)t^{\frac{p}{r}-1}x_k^{\frac{p}{r}-1}\left(t^{\frac{p}{r}} + \sum_{i=2}^n x_i^{\frac{p}{r}}\right)^{r-2} - p(p-1)(t - x_k)^{p-2} \\ &= p(p-1)\left((t + \sum_{i=2}^n x_i)^{p-2} - (t - x_k)^{p-2}\right) - \frac{p^2}{r}(r-1)t^{\frac{p}{r}-1}x_k^{\frac{p}{r}-1}\left(t^{\frac{p}{r}} + \sum_{i=2}^n x_i^{\frac{p}{r}}\right)^{r-2}. \end{aligned}$$

Since $p - 2 < 0$, we have

$$G'_{x_k}(x_2, \cdots, x_n) \le 0.$$

Hence, $G(x_2, \cdots, x_n)$ is a decreasing function for $x_k (k = 2, \cdots, n)$ on $[0, t]$. Hence, we have

$$F_1'(t) = G(b_2, \cdots, b_n) \le G(0, \cdots, 0) = pt^{p-1} - pnt^{p-1} + p(n-1)t^{p-1} = 0.$$

Which implies $F_1(t)$ is a decreasing function on $[b_2, +\infty]$. Hence, we have

$$F_1(b_1) \le F_1(b_2).$$

By similar method in the proof of theorem 1, we obtain

$$F_1(b_1) \le F_{n-1}(b_n),$$

$$F_{n-1}(t) = [(n-1)t + b_n]^p - n\left[(n-1)t^{\frac{p}{r}} + b_n^{\frac{p}{r}}\right]^r + (n-1)(t-b_n)^p.$$

Which implies $F_1(b_1) \le b_n^p(n^p - n^{1+r}) \le 0$, this is the inequality (2). The theorem is proved.

Theorem 3 *Let $a_i (1 \le i \le n)$ be nonnegative real numbers, and $1 \le p \le 2$. Then*

$$M_1^2(a_i : 1 \le i \le n) \le M_p^2(a_i : 1 \le i \le n) - \frac{p-1}{2} M_2^2(|a_i - a_j| : 1 \le i, j \le n),$$

that is

$$\left(\frac{1}{n}\sum_{i=1}^n a_i\right)^2 \le \left(\frac{1}{n}\sum_{i=1}^n a_i^p\right)^{\frac{2}{p}} - \frac{p-1}{2n^2}\sum_{i=1}^n\sum_{j=1}^n |a_i - a_j|^2. \tag{3}$$

Proof By similar method in the proof of theorem 1, the inequality (3) can be obtained.

Theorem 4 *Let nonnegative function $f(x)$ be integrable on $[a, b]$, and $1 \le p \le 2$. Then*

$$\begin{aligned} M_1^2(f(x) : a \le x \le b) \le\ & M_p^2(f(x) : a \le x \le b) \\ & -\frac{p-1}{2} M_2^2(|f(x) - f(y)| : a \le x, y \le b), \end{aligned}$$

that is

$$\begin{aligned} \left(\frac{1}{b-a}\int_a^b f(x)dx\right)^2 \le\ & \left(\frac{1}{b-a}\int_a^b f^p(x)dx\right)^{\frac{2}{p}} \\ & -\frac{p-1}{2(b-a)^2}\int_a^b\int_a^b |f(x) - f(y)|dxdy. \end{aligned} \tag{4}$$

Proof By the theorem 3, the inequality (4) can be obtained.

Corollary 1 *Let $a_i (1 \le i \le n)$ be nonnegative real numbers.*

(i) If $p \ge 2$, then we have

$$\left(\sum_{i=1}^n a_i\right)^p \le n^{p-1}\sum_{i=1}^n a_i^p - \sum_{1\le i<j\le n} |a_i - a_j|^p. \tag{5}$$

(ii) If $1 \le p \le 2$, then we have

$$\left(\sum_{i=1}^{n} a_i\right)^p \le n \sum_{i=1}^{n} a_i^p - \sum_{1 \le i < j \le n} |a_i - a_j|^p. \tag{6}$$

Proof Let $r = 1$ in (1) and (2), and (5) and (6) can be obtained respectively.

References

[1] Kuang Jichang, *Applied Inequalities*, China Shandong Science and Technology press, Jinan, 2004.