APPROXIMATION OF THE LAMBERT W FUNCTION

MEHDI HASSANI

Abstract. In this short note, we approximate the Lambert W function $W(x)$, defined by $W(x)e^{W(x)} = x$ for $x \geq -e^{-1}$. We show that

$$\log x - \log\log x < W(x) < \log x,$$

which the left hand side holds true for $x > 41.19$ and the right hand side holds true for $x > e$.

Definition. The Lambert W function $W(x)$, defined by $W(x)e^{W(x)} = x$ for $x \geq -e^{-1}$. For $-e^{-1} \leq x < 0$, there are two possible values of $W(x)$, which we takes such values that aren't less than -1. The history of the function goes back to J. H. Lambert (1728-1777). For more detailed definition of W as a complex variable function, historical background and various applications of it in Mathematics and Physics, see [1].

Some Elementary Properties. It is easy to see that $W(-e^{-1}) = -1$, $W(0) = 0$ and $W(e) = 1$. Also, for $x > 0$, since $W(x)e^{W(x)} = x > 0$ and $e^{W(x)} > 0$, we have $W(x) > 0$. About derivation, an easy calculation yields that

$$\frac{d}{dx}W(x) = \frac{W(x)}{x(1 + W(x))}.$$

So, $x\frac{d}{dx}W(x) > 0$ holds true for $x > 0$ and consequently $W(x)$ is strictly increasing for $x > 0$ (and also for $-e^{-1} \leq x \leq 0$, but not by this reason). Specially, it yields that $W(x) > 0$ for $x > 0$.

Upper Bound. Let $U(x) = \log x - W(x)$. Easily, $\frac{d}{dx}U(x) = \frac{1}{x(W(x)+1)} > 0$ for $x > 0$. Thus, $U(x) > U(e) = \log e - W(e) = 0$ holds for $x > e$. Therefore, we have:

$$W(x) < \log x \qquad (x > e).$$

Lower Bound. Let $L(x) = W(x) - \log x + \log\log x$. Similar to obtained upper bound, for $x > 41.19$ we have $1 + W(x) < \log x$. So, for $x > 41.19$, we obtain $\frac{d}{dx}L(x) = \frac{1+W(x)-\log x}{x\log x(W(x)+1)} < 0$. Thus, $L(x) > \lim_{x\to+\infty} L(x) = 0$. Therefore, we have:

$$\log x - \log\log x < W(x) \qquad (x > 41.19).$$

Expansion and More Careful Bounds. The following expansion holds true both at 0 and infinity:

$$W(x) = \log x - \log\log x + \sum_{k=0}^{\infty}\sum_{m=1}^{\infty} c_{km}\frac{(\log\log x)^m}{(\log x)^{k+m}},$$

where $c_{km} = \frac{(-1)^k}{m!}S[k+m, k+1]$, where $S[k+m, k+1]$ is Stirling cycle number [1]. The series in above expansion being to be absolutely convergent and it can be

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rearranged into the form:

$$W(x) = L_1 - L_2 + \frac{L_2}{L_1} + \frac{L_2(L_2-2)}{2L_1^2} + \frac{L_2(2L_2^2 - 9L_2 + 6)}{6L_1^3} + O\left(\left(\frac{L_2}{L_1}\right)^4\right),$$

where $L_1 = \log x$ and $L_2 = \log\log x$. Considering this expansion, it is possible to state and proof some more careful bounds, which isn't the aim of this work and we leave it as an open work.

References

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Institute for Advanced, Studies in Basic Sciences, P.O. Box 45195-1159, Zanjan, Iran.
E-mail address: mmhassany@srttu.edu