Geometrically Convex Function and Estimation of Remainder Terms in Taylor Series Expansion of some Functions*

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December 21, 2005

Abstract

In this paper, two integral inequalities of geometrically convex functions are proved. For their application, estimation formulas of remainder terms in Taylor series expansion of e^{-x}, $sinx$ and $cosx$ are given.

Key words: inequality, geometrically convex function, definite integral, remainder term.

1 Introduction and Definition

The paper first proves two integral inequality of geometrically concave functions, then obtains the estimation formulas of remainder terms in Taylor series expansion of e^{-x}, $sint$ and $cost$, where $x \in (0, +\infty)$ and $t \in \left(0, \frac{\pi}{2}\right)$.

In [1] [2] [3], the authors obtained some relative definitions of geometrically convex function.

DEFINITION 1. *Let $f : I \subseteq (0, +\infty) \to (0, +\infty)$ be a continuous function. then f is called a geometrically convex function on I, if there exists $n \geq 2$, such that one of the following inequalities holds for any $x_1, x_2, \cdots, x_n \in I$ and $\alpha, \beta, \lambda_1, \lambda_2, \cdots, \lambda_n > 0$ with*

$$f\left(\sqrt{x_1 x_2}\right) \leq \sqrt{f(x_1) f(x_2)}. \tag{1}$$

$$f\left(x_1^{\alpha} x_2^{\beta}\right) \leq f^{\alpha}(x_1) f^{\beta}(x_2). \tag{2}$$

$$f\left(\sqrt[n]{\prod_{i=1}^{n} x_i}\right) \leq \sqrt[n]{\prod_{i=1}^{n} f(x_i)}. \tag{3}$$

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*Mathematics Subject Classifications: 26D15, 26D10.

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$$f\left(\prod_{i=1}^{n} x_i^{\lambda_i}\right) \le \prod_{i=1}^{n} f^{\lambda_i}(x_i). \tag{4}$$

And f is called a geometrically concave function on I if one of inequalities (1)-(4) is inverse.

LEMMA 1. *([5]) Let $(a,b) \subseteq (0,+\infty)$, $f:(a,b) \to (0,+\infty)$ be twice differentiable. Then f is a geometrically convex function if, and only if*

$$x\left[f(x) f''(x) - (f'(x))^2\right] + f(x) f'(x) \ge 0 \tag{5}$$

hold for every $x \in (a,b)$. Meanwhile f is a geometrically concave function if, and only if inequality (5) is inverse.

LEMMA 2. *([2][5]) Let $0 < a < b$, $f:[a,b] \to (0,+\infty)$ be a geometrically convex(concave) function, and $g(x) = \ln f(e^x)$, $x \in [\ln a, \ln b]$. Then g is a convex(concave) function.*

For convex(concave) functions, unilateral Derivative exists. Lemma 2 tells us that unilateral Derivative of geometrically convex(concave) functions exist.

LEMMA 3. *([8]) Let $0 \le a < b$, function f be continuous on $[a,b]$, be a geometrically concave function on $(a,b]$, and $g(x) = \int_a^x f(t)\,dt, x \in (a,b]$. Then g is a geometrically concave function on $(a,b]$.*

2 Two integral inequalities

THEOREM 1. *Let $0 < a < b$, $f:[a,b] \to (0,+\infty)$ be a geometrically convex function. If $f(a) + af'_+(a) \ne 0$,*

$$\int_a^b f(x)\,dx \ge \frac{(f(a))^2}{\left(f(a) + af'_+(a)\right) a^{\frac{af'_+(a)}{f(a)}}} \cdot \left(b^{1+\frac{af'_+(a)}{f(a)}} - a^{1+\frac{af'_+(a)}{f(a)}}\right). \tag{6}$$

and if $f(b) + bf'_1(b) \ne 0$,

$$\int_a^b f(x)\,dx \ge \frac{(f(b))^2}{\left(f(b) + bf'_-(b)\right) b^{\frac{bf'_-(b)}{f(b)}}} \cdot \left(b^{1+\frac{bf'_-(b)}{f(b)}} - a^{1+\frac{bf'_-(b)}{f(b)}}\right). \tag{7}$$

The equality occurs only when f is a power function or constant function. Meanwhile inequality (6)(7) are inverse if f is a geometrically concave function.

Proof. Let f be a geometrically convex function.

$$\lim_{c \to a+0} \frac{\ln f(c) - \ln f(a)}{\ln c - \ln a} = \lim_{c \to a+0} \frac{\ln f\left(e^{\ln c}\right) - \ln f\left(e^{\ln a}\right)}{\ln c - \ln a}$$

$$= \left(\ln f\left(e^t\right)\right)'_+\Big|_{t=\ln a} = \frac{e^t f'_+\left(e^t\right)}{f\left(e^t\right)}\Bigg|_{t=\ln a} = \frac{a \cdot f'_+(a)}{f(a)} \ne -1.$$

Then we can choose $c \in (a,b)$ such that $\frac{\ln f(c)-\ln f(a)}{\ln c-\ln a} \neq -1$. Suppose $x \in [c,b]$, $t = \frac{\ln x-\ln c}{\ln x-\ln a}$. Hence $0 \leq t < 1$, $\left(\frac{x}{a}\right)^t = \frac{x}{c}$, $c = a^t x^{1-t}$ and $f(c) = f\left(a^t x^{1-t}\right)$. With respect to the definition 1, we have

$$f(c) \leq (f(a))^t (f(x))^{1-t}, \tag{8}$$

$$(f(c))^{\frac{1}{1-t}} (f(a))^{-\frac{t}{1-t}} \leq f(x),$$

$$(f(c))^{\frac{\ln x-\ln a}{\ln c-\ln a}} (f(a))^{-\frac{\ln x-\ln c}{\ln c-\ln a}} \leq f(x).$$

Hence

$$\int_c^b f(x)\,dx \geq \int_c^b (f(c))^{\frac{\ln x-\ln a}{\ln c-\ln a}} (f(a))^{-\frac{\ln x-\ln c}{\ln c-\ln a}}\,dx$$

$$= (f(c))^{-\frac{\ln a}{\ln c-\ln a}} (f(a))^{\frac{\ln c}{\ln c-\ln a}} \int_c^b (f(c))^{\frac{\ln x}{\ln c-\ln a}} (f(a))^{-\frac{\ln x}{\ln c-\ln a}}\,dx.$$

Let $\ln x = u, u = e^x$, we have

$$\int_c^b f(x)\,dx \geq (f(c))^{-\frac{\ln a}{\ln c-\ln a}} (f(a))^{\frac{\ln c}{\ln c-\ln a}} \int_{\ln c}^{\ln b} (f(c))^{\frac{u}{\ln c-\ln a}} (f(a))^{-\frac{u}{\ln c-\ln a}}\,d(e^u)$$

$$= (f(c))^{-\frac{\ln a}{\ln c-\ln a}} \cdot (f(a))^{\frac{\ln c}{\ln c-\ln a}} \int_{\ln c}^{\ln b} \left[e (f(c))^{\frac{1}{\ln c-\ln a}} \cdot (f(a))^{-\frac{1}{\ln c-\ln a}}\right]^u du$$

$$= (f(c))^{-\frac{\ln a}{\ln c-\ln a}} \cdot (f(a))^{\frac{\ln c}{\ln c-\ln a}}$$

$$\cdot \frac{b (f(c))^{\frac{\ln b}{\ln c-\ln a}} (f(a))^{-\frac{\ln b}{\ln c-\ln a}} - c (f(c))^{\frac{\ln c}{\ln c-\ln a}} (f(a))^{-\frac{\ln c}{\ln c-\ln a}}}{1 + \frac{\ln f(c)}{\ln c-\ln a} - \frac{\ln f(a)}{\ln c-\ln a}}$$

$$= \frac{b (f(c))^{\frac{\ln b-\ln a}{\ln c-\ln a}} (f(a))^{\frac{\ln c-\ln b}{\ln c-\ln a}} - c \cdot f(c)}{1 + \frac{\ln f(c)-\ln f(a)}{\ln c-\ln a}}$$

$$= f(c) \cdot \frac{b \left(\frac{f(c)}{f(a)}\right)^{\frac{\ln b-\ln c}{\ln c-\ln a}} - c}{1 + \frac{\ln f(c)-\ln f(a)}{\ln c-\ln a}}$$

$$= f(c) \cdot \frac{b \cdot \exp\left(\frac{\ln b-\ln c}{\ln c-\ln a} \cdot (\ln f(c) - \ln f(a))\right) - c}{1 + \frac{\ln f(c)-\ln f(a)}{\ln c-\ln a}}. \tag{9}$$

Let $c \to a+0$ in inequality (9), we get

$$\int_a^b f(x)\,dx \geq f(a) \frac{b \cdot \exp\left(a (\ln b - \ln a) \frac{f'_+(a)}{f(a)}\right) - a}{1 + \frac{a f'_+(a)}{f(a)}}$$

$$= (f(a))^2 \frac{b \cdot \left(\frac{b}{a}\right)^{\frac{a f'_+(a)}{f(a)}} - a}{f(a) + a \cdot f'_+(a)}$$

$$= \frac{(f(a))^2}{\left(f(a) + af'_+(a)\right) a^{\frac{af'_+(a)}{f(a)}}} \cdot \left(b^{1+\frac{af'_+(a)}{f(a)}} - a^{1+\frac{af'_+(a)}{f(a)}}\right).$$

Equality (6) occurs only when equality (8) occurs, f is a power function or constant function.

Analogously we can proof inequality (7).

The proof of Theorem 1 is completed. □

THEOREM 2. *Let $0 < a < b$, $f : [a,b] \to (0,+\infty)$ be a geometrically convex function. Then*

$$\int_a^b f(x)\,dx \le \begin{cases} \frac{bf(b)-af(a)}{\ln(bf(b))-\ln(af(a))} \cdot \ln\frac{b}{a}, & af(a) \ne bf(b), \\ af(a) \cdot \ln\frac{b}{a}, & af(a) = bf(b). \end{cases} \tag{10}$$

The equality occurs only when f is a power function or constant function. Meanwhile inequality (10) *is inverse if f is a geometrically concave function.*

Proof. Let f is a geometrically convex function. For $\int_a^b f(x)\,dx$, Let $\alpha = \log_{\frac{b}{a}} \frac{x}{a}$, $x = a^{1-\alpha} b^{\alpha}$. Then

$$\int_a^b f(x)\,dx = \int_0^1 f\left(a^{1-\alpha} b^{\alpha}\right) \cdot a^{1-\alpha} b^{\alpha} \cdot \ln\frac{b}{a} d\alpha.$$

With respect to the definition 1, if $af(a) \ne bf(b)$, we have

$$\int_a^b f(x)\,dx \le \int_0^1 f^{1-\alpha}(a) \cdot f^{\alpha}(b) \cdot a^{1-\alpha} b^{\alpha} \cdot \ln\frac{b}{a} d\alpha$$

$$= af(a) \cdot \ln\frac{b}{a} \cdot \int_0^1 \left(\frac{bf(b)}{af(a)}\right)^{\alpha} d\alpha = \frac{af(a)}{\ln\left(\frac{bf(b)}{af(a)}\right)} \cdot \ln\frac{b}{a} \cdot \left(\frac{bf(b)}{af(a)}\right)^{\alpha}\Bigg|_0^1$$

$$= \frac{af(a)}{\ln(bf(b) - af(a))} \cdot \ln\frac{b}{a} \cdot \left[\frac{bf(b)}{af(a)} - 1\right] = \frac{bf(b) - af(a)}{\ln(bf(b)) - \ln(af(a))} \cdot \ln\frac{b}{a}.$$

If $af(a) = bf(b)$,

$$\int_a^b f(x)\,dx \le af(a) \ln\frac{b}{a}$$

is obvious. □

3 Estimation Formula of e^{-x}

THEOREM 3. *Let $x > 0, n \in N, n \ge 1$, and $T_n(x) = e^{-x} - 1 + x - \frac{x^2}{2!} + \cdots + (-1)^{n-1} \frac{x^n}{n!}$.*

(1)If $0 < x < \frac{n+2}{n+1}$. Then

$$\frac{2(n+1)}{n + x + \sqrt{(n+2+x)^2 - 4x}} \cdot \frac{x^{n+1}}{(n+1)!} \le |T_n(x)| \le \frac{n+2}{n+2+x} \cdot \frac{x^{n+1}}{(n+1)!}. \tag{11}$$

(2)If $x \geq \frac{n+2}{n+1}$. Then

$$\frac{2(n+1)}{n+x+\sqrt{(n+2+x)^2-4x}}\cdot\frac{x^{n+1}}{(n+1)!} \leq |T_n(x)| \leq \frac{2(n+1)}{n+1+x+\sqrt{(x+n+1)^2-4x}}\cdot\frac{x^{n+1}}{(n+1)!}. \tag{12}$$

Proof. If n is an odd number, let $n = 2m-1, m \in N, m \geq 1$. If n is an even number, let $n = 2m, m \in N, m \geq 1$. According to Taylor series expansion of e^{-x}, we have

$$T_{2m-1}(x) = e^{-x} - 1 + x - \frac{x^2}{2!} + \cdots + \frac{x^{2m-1}}{(2m-1)!} \geq 0,$$

$$T_{2m}(x) = e^{-x} - 1 + x - \frac{x^2}{2!} + \cdots - \frac{x^{2m}}{(2m)!} \leq 0,$$

$$-T_{2m}(x) = -e^{-x} + 1 - x + \frac{x^2}{2!} + \cdots + \frac{x^{2m}}{(2m)!} \geq 0.$$

Suppose $f(x) = e^{-x}, x \in (0, +\infty)$, then f is a geometrically concave function according to Lemma 1. Further since

$$\int_0^x e^{-t}dt = 1 - e^{-x}, \quad \int_0^x \left(1 - e^{-t}\right)dt = e^{-x} - 1 + x,$$

$$\int_0^x \left(e^{-t} - 1 + x\right)dt = -e^{-x} + 1 - x + \frac{x^2}{2}, \quad \cdots.$$

T_{2m-1} and $-T_{2m}$ are both geometrically concave functions according to Lemma 3.

Suppose $0 < a < x$, according to Theorem 1, we get

$$\int_a^x T_{2m-1}(x)dx \leq \frac{T_{2m-1}(x)}{(1+\eta)x^{\eta}} \cdot \left(x^{1+\eta} - a^{1+\eta}\right), \tag{13}$$

where $\eta = \frac{xT'_{2m-1}(x)}{T_{2m-1}(x)} = \frac{x\left(-e^{-x}+1-x+\cdots+\frac{x^{2m-2}}{(2m-2)!}\right)}{T_{2m-1}(x)} > 0$. Let $a \to 0+$ in (13),

$$-e^{-x} + 1 - x + \frac{x^2}{2!} + \cdots + \frac{x^{2m}}{(2m)!} \leq \frac{xT_{2m-1}(x)}{1 + \frac{xT'_{2m-1}(x)}{T_{2m-1}(x)}},$$

$$-T_{2m-1}(x) + \frac{x^{2m}}{(2m)!} \leq \frac{xT^2_{2m-1}(x)}{T_{2m-1}(x) + x\left(-e^{-x} + 1 - x + \cdots + \frac{x^{2m-2}}{(2m-2)!}\right)},$$

$$-T_{2m-1}(x) + \frac{x^{2m}}{(2m)!} \leq \frac{xT^2_{2m-1}(x)}{T_{2m-1}(x) + x\left(-T_{2m-1}(x) + \frac{x^{2m-1}}{(2m-1)!}\right)},$$

$$(x-1)T^2_{2m-1}(x) + \left(-\frac{x^{2m}}{(2m-1)!} + \frac{x^{2m}}{(2m)!} - \frac{x^{2m+1}}{(2m)!}\right)T_{2m-1}(x) + \frac{x^{4m}}{(2m)!(2m-1)!} \leq xT^2_{2m-1}(x),$$

$$T_{2m-1}^2(x)+\frac{(2m-1+x)x^{2m}}{(2m)!}T_{2m-1}(x)-\frac{x^{4m}}{(2m)!(2m-1)!}\geq 0,$$

$$\begin{aligned}
T_{2m-1}(x) &\geq \frac{-\frac{(2m-1+x)x^{2m}}{(2m)!}+\sqrt{\frac{(2m-1+x)^2x^{4m}}{((2m)!)^2}+\frac{4x^{4m}}{(2m)!(2m-1)!}}}{2}\\
&=\frac{-(2m-1+x)+\sqrt{(2m-1+x)^2+8m}}{2}\cdot\frac{x^{2m}}{(2m)!}\\
&=\frac{4m}{\sqrt{(2m-1+x)^2+8m}+2m-1+x}\cdot\frac{x^{2m}}{(2m)!}\\
&=\frac{4m}{\sqrt{(2m+1+x)^2-4x}+2m-1+x}\cdot\frac{x^{2m}}{(2m)!}. \qquad (14)
\end{aligned}$$

Since $-T_{2m}$ also is a geometrically concave function, by Theorem 1 analogously we have

$$-T_{2m}(x)\geq\frac{4m+2}{2m+x+\sqrt{(2m+2+x)^2-4x}}\cdot\frac{x^{2m+1}}{(2m+1)!}. \qquad (15)$$

Let $0<a<x$, according to Theorem 2, we get

$$\int_a^x T_{2m-1}(t)\,dt\geq\frac{xT_{2m-1}(x)-aT_{2m-1}(a)}{\ln(xT_{2m-1}(x))-\ln(aT_{2m-1}(a))}\cdot\ln\frac{x}{a},$$

$$\int_a^x T_{2m-1}(t)\,dt\geq(xT_{2m-1}(x)-aT_{2m-1}(a))\cdot\frac{\ln x-\ln a}{\ln(xT_{2m-1}(x))-\ln(aT_{2m-1}(a))}.$$

Let $a\to 0+$, we note the L'Hospital Rule,

$$\int_0^x T_{2m-1}(t)\,dt\geq xT_{2m-1}(x)\cdot\lim_{a\to 0+}\frac{-1/a}{-\frac{T_{2m-1}(a)+aT'_{2m-1}(a)}{T_{2m-1}(a)}},$$

$$-e^{-x}+1-x+\frac{x^2}{2!}+\cdots+\frac{x^{2m}}{(2m)!}\geq\frac{xT_{2m-1}(x)}{1+\lim\limits_{a\to 0+}\frac{aT'_{2m-1}(a)}{T_{2m-1}(a)}},$$

$$-T_{2m-1}(x)+\frac{x^{2m}}{(2m)!}\geq\frac{xT_{2m-1}(x)}{2+\lim\limits_{a\to 0+}\frac{aT''_{2m-1}(a)}{T'_{2m-1}(a)}},$$

$$-T_{2m-1}(x)+\frac{x^{2m}}{(2m)!}\geq\frac{xT_{2m-1}(x)}{2m+\lim\limits_{a\to 0+}\frac{ae^{-a}}{1-e^{-a}}},$$

$$-T_{2m-1}(x)+\frac{x^{2m}}{(2m)!}\geq\frac{xT_{2m-1}(x)}{2m+1},$$

$$\frac{(2m+1)x^{2m}}{(2m)!}\geq(2m+1+x)T_{2m-1}(x),$$

$$T_{2m-1}(x) \le \frac{(2m+1)}{(2m+1+x)} \cdot \frac{x^{2m}}{(2m)!}. \tag{16}$$

Also, for $-T_{2m}(x)$, similarly we can get

$$-T_{2m}(x) \le \frac{(2m+2)}{(2m+2+x)} \cdot \frac{x^{2m+1}}{(2m+1)!}. \tag{17}$$

We omit the details.

Since $T_{2m-1}(x)$ is a geometrically concave function, according to Lemma 1 we have

$$x\left(T_{2m-1}(x) \cdot T''_{2m-1}(x) - \left(T'_{2m-1}(x)\right)^2\right) + T_{2m-1}(x) \cdot T'_{2m-1}(x) \le 0. \tag{18}$$

Meanwhile,

$$T'_{2m-1}(x) = -T_{2m-1}(x) + \frac{x^{2m-1}}{(2m-1)!},$$

$$T''_{2m-1}(x) = T_{2m-1}(x) + \frac{x^{2m-2}}{(2m-2)!} - \frac{x^{2m-1}}{(2m-1)!}.$$

So

$$-T^2_{2m-1}(x) + \frac{2mx^{2m-1} + x^{2m}}{(2m-1)!} T_{2m-1}(x) - \frac{x^{4m-1}}{((2m-1)!)^2} \le 0,$$

$$T^2_{2m-1}(x) - \frac{2mx^{2m-1} + x^{2n}}{(2m-1)!} T_{2m-1}(x) + \frac{x^{4m-1}}{((2m-1)!)^2} \ge 0,$$

$$T_{2m-1} \le \frac{2m + x - \sqrt{(x+2m)^2 - 4x}}{2} \cdot \frac{x^{2m-1}}{(2m-1)!}$$

$$= \frac{2x}{2m + x + \sqrt{(2m+x)^2 - 4x}} \cdot \frac{x^{2m-1}}{(2m-1)!} = \frac{4m}{2m + x + \sqrt{(2m+x)^2 - 4x}} \cdot \frac{x^{2m}}{(2m)!}. \tag{19}$$

Also, for $-T_{2m}(x)$, after some similarly calculus we can get

$$-T_{2m}(x) \le \frac{4m+2}{2m+1+x+\sqrt{(2m+1+x)^2 - 4x}} \cdot \frac{x^{2m+1}}{(2m+1)!}, \tag{20}$$

If $0 < x \le \frac{2n+1}{2n}$, we can find inequality (16) is stronger than inequality (19). If $x \ge \frac{2n+1}{2n}$, inequality (19) is stronger than inequality (16). If $0 < x \le \frac{2n+2}{2n+1}$, we can find inequality (17) is stronger than inequality (20). If $x \ge \frac{2n+2}{2n+1}$, we can find inequality (20) is stronger than inequality (17).

The proof of Theorem 3 is completed. □

4 Estimation Formula of $\sin x$ and $\cos x$

THEOREM 4. *Let $n \geq 1$, $x \in \left(0, \frac{\pi}{2}\right)$, $S_n(x) = \sin x - x + \frac{x^3}{3!} + \cdots + (-1)^n \frac{x^{2n-1}}{(2n-1)!}$ and $P_n(x) = \cos x - 1 + \frac{x^2}{2!} + \cdots + (-1)^{n+1} \frac{x^{2n}}{(2n)!}$. Then*

$$\frac{x^{2n+1}}{(2n+1)!} \cdot \frac{2n(2n+1)}{2n(2n+1)+x^2} \leq |S_n(x)| \leq \frac{x^{2n+1}}{(2n+1)!} \cdot \frac{(2n+2)(2n+3)}{(2n+2)(2n+3)+x^2}. \quad (21)$$

$$\frac{x^{2n+2}}{(2n+2)!} \cdot \frac{(2n+1)(2n+2)}{(2n+1)(2n+2)+x^2} \leq |P_n(x)| \leq \frac{x^{2n+2}}{(2n+2)!} \cdot \frac{(2n+3)(2n+4)}{(2n+3)(2n+4)+x^2}. \quad (22)$$

Proof. Let $x \in \left(0, \frac{\pi}{2}\right)$, and

$$f_1(x) = -\sin x + x - \frac{x^3}{3!} - \cdots + \frac{x^{4n-3}}{(4n-3)!}, \quad f_2(x) = \cos x - 1 + \frac{x^2}{2!} - \cdots + \frac{x^{4n-2}}{(4n-2)!},$$

$$f_3(x) = \sin x - x + \frac{x^3}{3!} - \cdots + \frac{x^{4n-1}}{(4n-1)!}, \quad f_4(x) = -\cos x + 1 - \frac{x^2}{2!} + \cdots + \frac{x^{4n}}{(4n)!},$$

$$f_5(x) = -\sin x + x - \frac{x^3}{3!} - \cdots + \frac{x^{4n+1}}{(4n+1)!}, f_6(x) = \cos x - 1 + \frac{x^2}{2!} - \cdots + \frac{x^{4n+2}}{(4n+2)!},$$

$$f_7(x) = \sin x - x + \frac{x^3}{3!} - \cdots + \frac{x^{4n+3}}{(4n+3)!}, \quad f_8(x) = -\cos x + 1 - \frac{x^2}{2!} - \cdots + \frac{x^{4n+4}}{(4n+4)!}.$$

We can find that $\sin : x \in \left(0, \frac{\pi}{2}\right) \to \sin x$ and $\cos : x \in \left(0, \frac{\pi}{2}\right) \to \cos x$ are geometrically concave functions according to Lemma 1([5]). By Lemma 2, the following functions all are geometrically concave functions on $\left(0, \frac{\pi}{2}\right)$,

$$\int_0^x \sin t dt = 1 - \cos x, \int_0^x (1 - \cos t)\, dt = x - \sin x,$$

$$\int_0^x (t - \sin t)\, dt = \cos x - 1 + \frac{x^2}{2}, \cdots.$$

So functions $f_i\ (i = 1, 2, \cdots, 8)$ also are geometrically concave functions on $\left(0, \frac{\pi}{2}\right)$. For $0 < a < x$ and $i = 1, \cdots, 7, 8$, Theorem 2 tells us that

$$\int_a^x f_i(t)\, dt \geq \frac{x f_i(x) - a f_i(a)}{\ln(x f_i(x)) - \ln(a f_i(a))} \cdot (\ln x - \ln a).$$

Let $a \to 0+$, have

$$\int_0^x f_i(t)\, dt \geq \frac{x f_i(x)}{1 + \lim\limits_{a \to 0+} \frac{a \cdot f_i'(a)}{f_i(a)}},$$

$$f_{i+1}(x) \geq \frac{x f_i(x)}{1 + \lim\limits_{a \to 0+} \frac{a \cdot f_i'(a)}{f_i(a)}}. \quad (23)$$

With respect to L'Hospital-Rule, inequality (23) becomes

$$f_{i+1}(x) \geq \frac{x f_i(x)}{4n+i+1}. \tag{24}$$

So

$$f_3(x) \geq \frac{x f_1(x)}{4n+1}, \quad f_2(x) \geq \frac{x f_1(x)}{4n},$$

$$f_4(x) \geq \frac{x f_3(x)}{4n+2}, \quad f_5(x) \geq \frac{x f_4(x)}{4n+3}.$$

$$f_3(x) \geq \frac{x f_2(x)}{4n+1} \geq \frac{x^2 f_1(x)}{4n(4n+1)} = \frac{x^2}{4n(4n+1)} \left(-f_3(x) + \frac{x^{4n-1}}{(4n-1)!} \right),$$

$$4n(4n+1) f_3(x) \geq -x^2 f_3(x) + \frac{x^{4n+1}}{(4n-1)!},$$

$$f_3(x) \geq \frac{x^{4n+1}}{(4n+1)!} \cdot \frac{4n(4n+1)}{4n(4n+1)+x^2}. \tag{25}$$

Furthermore

$$\frac{x f_3(x)}{4n+2} \leq f_4(x) \leq \frac{4n+3}{x} f_5(x) = \frac{4n+3}{x} \left(-f_3(x) + \frac{x^{4n+1}}{(4n+1)!} \right),$$

$$x^2 f_3(x) \leq -(4n+2)(4n+3) f_3(x) + (4n+2)(4n+3) \frac{x^{4n+1}}{(4n+1)!},$$

$$f_3(x) \leq \frac{x^{4n+1}}{(4n+1)!} \cdot \frac{(4n+2)(4n+3)}{(4n+2)(4n+3)+x^2}. \tag{26}$$

According to (25) and (26),

$$\frac{x^{4n+1}}{(4n+1)!} \cdot \frac{4n(4n+1)}{4n(4n+1)+x^2} \leq f_3(x) \leq \frac{x^{4n+1}}{(4n+1)!} \cdot \frac{(4n+2)(4n+3)}{(4n+2)(4n+3)+x^2}. \tag{27}$$

Analogously, after some calculus we can get

$$\frac{x^{4n+2}}{(4n+2)!} \cdot \frac{(4n+1)(4n+2)}{(4n+1)(4n+2)+x^2} \leq f_4(x) \leq \frac{x^{4n+2}}{(4n+2)!} \cdot \frac{(4n+3)(4n+4)}{(4n+3)(4n+4)+x^2}, \tag{28}$$

$$\frac{x^{4n+3}}{(4n+3)!} \cdot \frac{(4n+2)(4n+3)}{(4n+2)(4n+3)+x^2} \leq f_5(x) \leq \frac{x^{4n+3}}{(4n+3)!} \cdot \frac{(4n+4)(4n+5)}{(4n+4)(4n+5)+x^2}, \tag{29}$$

$$\frac{x^{4n+4}}{(4n+4)!} \cdot \frac{(4n+3)(4n+4)}{(4n+3)(4n+4)+x^2} \leq f_6(x) \leq \frac{x^{4n+4}}{(4n+4)!} \cdot \frac{(4n+5)(4n+6)}{(4n+5)(4n+6)+x^2}. \tag{30}$$

The proof of Theorem 4 is completed.

REMARK 1. *We shall present other applications of Theorem 1 and Theorem 2 in another paper, also includes estimation formula of remainder terms in Taylor series expansion of* e^x.

□

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