APPROXIMATION OF p_n BY H_n

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Abstract. In this paper we introduce some bounds including $H_n = \sum_{k=1}^{n} \frac{1}{k}$, for p_n, n^{th} prime number. Then we observe that the Prime Number Theorem is equivalent with $p_n \sim nH_n$, when n tends to infinity.

1. Introduction

As usual, let p_n be the n^{th} prime. According to the Prime Number Theorem (PNT) [3], we know that:

$$p_n = n \log n + o(n \log n) \qquad (n \to \infty). \tag{1.1}$$

Also, we know that [1], if $H_n = \sum_{k=1}^{n} \frac{1}{k}$, then:

$$H_n = \log n + O(1) \qquad (n \to \infty). \tag{1.2}$$

So, considering (1.1) and (1.2), we obtain:

$$p_n = n(H_n + O(1)) + o(n \log n) = nH_n + o(n \log n) \qquad (n \to \infty).$$

Therefore, comparing p_n and nH_n seems to be a nice problem. To consider this problem, we need some bounds concerning p_n and H_n, which we recall them from literatures. About p_n, we have the following bounds [4]:

(1.3)

$$n \log n + n \log_2 n - n + n\frac{\log_2 n - 2.25}{\log n} \leq p_n \leq n \log n + n \log_2 n - n + n\frac{\log_2 n - 1.8}{\log n},$$

which left hand side of it holds true for every $n \geq 2$ and the right hand side of it holds true for every $n \geq 27076$, $\log_2 n$ means $\log \log n$ and base of all logarithms is e. Also, for H_n we have the following bounds [2]:

$$\gamma + \log(n + 0.5) < H_n \leq \gamma + \log(n - 1 + e^{1-\gamma}) \qquad (n \geq 1), \tag{1.4}$$

where γ is Euler constant. In this note, we search some bounds of the form $nH_n + e(n)$, which we will find $e(n)$ in both cases lower and upper bounds.

2. Inequalities of the form nH_n

Consider the following inequality:

$$nH_n + a(n) \leq p_n \leq nH_n + b(n). \tag{2.1}$$

Here, we try to find some suitable functions $a(n)$ and $b(n)$, such that (2.1) holds true.

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Upper Bound. To find above mentioned upper bound, considering (1.4), we have:

$$n\left(\gamma+\log(n+0.5)\right) < nH_n \leq n\left(\gamma+\log(n-1+e^{1-\gamma})\right). \tag{2.2}$$

Using left hand side of above inequality, we have:

$$n\left(\gamma+\log(n+0.5)\right)+b(n) \leq nH_n+b(n),$$

and considering this inequality, with the right hand side of (1.3), we must have:

$$n\log n+n\log_2 n-n+n\frac{\log_2 n-1.8}{\log n} \leq n\left(\gamma+\log(n+0.5)\right)+b(n),$$

or equivalently,

$$n\log n+n\log_2 n-n+n\frac{\log_2 n-1.8}{\log n}-n\left(\gamma+\log(n+0.5)\right) \leq b(n),$$

Since, $b(n)$ is going to appears in upper bound for p_n , the best possible case is:

$$b(n)=n\log_2 n-n(1+\gamma)+n\left(\frac{\log_2 n-1.8}{\log n}-\log\left(1+\frac{0.5}{n}\right)\right). \tag{2.3}$$

Thus, we have:

$$p_n \leq nH_n+n\log_2 n-n(1+\gamma)+n\left(\frac{\log_2 n-1.8}{\log n}-\log\left(1+\frac{0.5}{n}\right)\right), \tag{2.4}$$

which holds for $n \geq 27076$.

Lower Bound. To find above mentioned lower bound, considering (2.2), we have:

$$nH_n+a(n) \leq n\gamma+n\log(n-1+e^{1-\gamma})+a(n).$$

Considering this inequality with the left hand side of (2.1) and the left hand side of (1.3), we must have:

$$n\gamma+n\log(n-1+e^{1-\gamma})+a(n) \leq n\log n+n\log_2 n-n+n\frac{\log_2 n-2.25}{\log n}.$$

Since, we want to find the maximum lower bound in the left hand side of (2.1), the best possible choice for $a(n)$, is:

$$a(n)=n\log_2 n-n(1+\gamma)+n\left(\frac{\log_2 n-2.25}{\log n}-\log\left(1+\frac{e^{1-\gamma}-1}{n}\right)\right).$$

So, we have:

$$nH_n+n\log_2 n-n(1+\gamma)+n\left(\frac{\log_2 n-2.25}{\log n}-\log\left(1+\frac{e^{1-\gamma}-1}{n}\right)\right) \leq p_n, \tag{2.5}$$

which holds for $n \geq 2$. Therefore, considering (2.4) and (2.5), for every $n \geq 27076$, we obtain:

$$\left|p_n-\left(nH_n+n\log_2 n-n(1+\gamma)\right)\right| \leq n\left(\frac{\log_2 n-1.8}{\log n}-\log\left(1+\frac{0.5}{n}\right)\right). \tag{2.6}$$

3. An Equivalent for the PNT

Considering (2.6), we obtain:

$$p_n = nH_n + n\log_2 n - n(1+\gamma) + O\left(\frac{n\log_2 n}{\log n}\right),$$

which is a very strong form of an equivalent for the PNT. In fact we observe that the PNT holds if and only if $p_n \sim nH_n$, when n tends to infinity. To see this, first suppose that the PNT holds true. So, when n tends to infinity, we have:

$$p_n = n\log n + o(n\log n).$$

Considering this with $H_n \sim \log n$, we obtain:

$$\begin{aligned} p_n &= n(H_n + O(1)) + o(n\log n) \\ &= nH_n + O(n) + o(n\log n) \\ &= nH_n + o(n\log n) \\ &= nH_n + o(n(H_n + O(1))) \\ &= nH_n + o(nH_n) + o(O(n)) \\ &= nH_n + o(nH_n). \end{aligned}$$

Inversely, suppose $p_n \sim nH_n$, then:

$$p_n = nH_n + o(nH_n),$$

which considering this with $H_n \sim \log n$, we obtain:

$$\begin{aligned} p_n &= n(\log n + O(1)) + o(n(\log n + O(1))) \\ &= n\log n + O(n) + o(n\log n + O(n)) \\ &= n\log n + O(n) + o(n\log n) \\ &= n\log n + o(n\log n), \end{aligned}$$

and this is PNT.

References

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