

MONOTONIC PROPERTIES OF DIFFERENCES FOR REMAINDERS OF PSI FUNCTION

FENG QI, DA-WEI NIU, AND BAI-NI GUO

ABSTRACT. Let $\Lambda_{p,q}(x) = \lambda(px) - q\lambda(x)$ and $\Phi_{p,q}(x) = \phi(px) - q\phi(x)$ in $x \in (0, \infty)$ for $p > 0$ and $q \in \mathbb{R}$, where $\lambda(x) = \int_0^\infty \frac{t dt}{(t^2+x^2)(e^{2\pi t}-1)}$ and $\phi(x) = \int_0^\infty \frac{t dt}{(t^2+4x^2)(e^{\pi t}+1)}$ are related to $\psi(x)$ and $\psi(x + \frac{1}{2})$. In this article, some sufficient conditions on $p > 0$ and $q \in \mathbb{R}$ such that $\Lambda_{p,q}(x)$ and $\Phi_{p,q}(x)$ are monotonic in $x \in (0, \infty)$ are obtained. Moreover, as by-product, an inequality involving the exponential function is established.

1. INTRODUCTION

Recall [7, 11] that a function f is said to be completely monotonic on an interval I if f has derivatives of all orders and $0 \leq (-1)^k f^{(k)}(x) < \infty$ for all $k \geq 0$ on I . For our own convenience, the class of completely monotonic functions on I is denoted by $\mathcal{C}[I]$. The well known Bernstein's Theorem [11] states that $f \in \mathcal{C}[(0, \infty)]$ if and only if $f(x) = \int_0^\infty e^{-xt} d\mu(t)$, where $\mu(t)$ is a nonnegative measure on $[0, \infty)$ such that the integral converges for all $x > 0$. Note that a completely monotonic function in $(0, \infty)$ which is non-identically zero cannot vanish at any point in $(0, \infty)$, see [7, 8] and the references therein.

The noted Binet's formula (see [9] and [10, p. 103]) states that for $x > 0$,

$$\ln \Gamma(x) = \left(x - \frac{1}{2}\right) \ln x - x + \ln \sqrt{2\pi} + \theta(x), \quad (1)$$

where

$$\theta(x) = \int_0^\infty \left(\frac{t}{e^t - 1} - 1 + \frac{t}{2}\right) \frac{e^{-xt}}{t^2} dt \quad (2)$$

is called the remainder of Binet's formula (1).

2000 *Mathematics Subject Classification.* Primary 33B15; Secondary 26D07, 26D20.

Key words and phrases. Monotonicity, difference, psi function, remainder, inequality.

The first author was supported in part by the Science Foundation of Project for Fostering Innovation Talents at Universities of Henan Province, China.

This paper was typeset using $\mathcal{A}\mathcal{M}\mathcal{S}\text{-}\mathcal{L}\mathcal{A}\mathcal{T}\mathcal{E}\mathcal{X}$.

Let $p > 0$ and $q \in \mathbb{R}$ be real numbers and

$$h_{p,q}(x) = \theta(px) - q\theta(x) \quad (3)$$

in $(0, \infty)$.

It is clear that $h_{1,q}(x) \in \mathcal{C}[(0, \infty)]$ for $q \leq 1$ and $-h_{1,q}(x) \in \mathcal{C}[(0, \infty)]$ for $q \geq 1$.

Among other things, the following was proved in [1, 2].

Theorem A ([1, 2]). $h_{p,p}(x) \in \mathcal{C}[(0, \infty)]$ if $0 < p < 1$ and $-h_{p,p}(x) \in \mathcal{C}[(0, \infty)]$ if $p > 1$.

As a further generalization of Theorem A above, the following conclusion was obtained recently.

Theorem B ([4]). $h_{p,q}(x) \in \mathcal{C}[(0, \infty)]$ if either $q \leq \frac{1}{p} \leq 1$ or $q \leq 1 \leq \frac{1}{p}$ and $-h_{p,q}(x) \in \mathcal{C}[(0, \infty)]$ if either $q \geq \frac{1}{p} \geq 1$ or $q \geq 1 \geq \frac{1}{p}$.

In [3, p. 892] and [6, p. 17], it is given that for $x > 0$,

$$\psi(x) = \ln x - \frac{1}{2x} - 2 \int_0^\infty \frac{t dt}{(t^2 + x^2)(e^{2\pi t} - 1)} \quad (4)$$

and

$$\psi\left(x + \frac{1}{2}\right) = \ln x + 2 \int_0^\infty \frac{t dt}{(t^2 + 4x^2)(e^{\pi t} + 1)}. \quad (5)$$

Let

$$\Lambda_{p,q}(x) = \lambda(px) - q\lambda(x) \quad (6)$$

and

$$\Phi_{p,q}(x) = \phi(px) - q\phi(x) \quad (7)$$

in $x \in (0, \infty)$ for $p > 0$ and $q \in \mathbb{R}$, where

$$\lambda(x) = \int_0^\infty \frac{t dt}{(t^2 + x^2)(e^{2\pi t} - 1)} \quad (8)$$

and

$$\phi(x) = \int_0^\infty \frac{t dt}{(t^2 + 4x^2)(e^{\pi t} + 1)}. \quad (9)$$

Our main results of this paper are given in the following theorems.

Theorem 1. *The function $\Lambda_{p,q}(x)$ is positive and decreasing in $x \in (0, \infty)$ if either $p \geq 1$ and $q \leq 0$ or $0 < p < 1$ and $pq \leq 1$; it is negative and increasing in $x \in (0, \infty)$ if $p \geq 1$ and $pq \geq 1$.*

The function $\Phi_{p,q}(x)$ is positive and decreasing in $x \in (0, \infty)$ if either $p \geq 1$ and $q \leq 0$ or $0 < p < 1$ and $q \leq 1$; it is negative and increasing in $x \in (0, \infty)$ if $p > 1$ and $q \geq 1$.

Theorem 2. *The function $\Lambda_{p,q}(x)$ is positive and decreasing in $x \in (0, \infty)$ for either $q \leq 0$ or $0 < q = \frac{1}{p^2} \leq 1$, it is negative and increasing in $x \in (0, \infty)$ for $\frac{1}{p^2} = q \geq 1$.*

The function $\Phi_{p,q}(x)$ is positive and decreasing in $x \in (0, \infty)$ for either $p^2q < 1$ and $q(p^2 - 1)[(1 + 3q)p^2 - 4] \leq 0$ or $p^2q = 1$ and $0 < q \leq 1$, it is negative and increasing in $x \in (0, \infty)$ for either $4 \leq p^2(1 + 3q) \leq 1 + 3q$ or $\frac{1}{p^2} = q \geq 1$.

As by-product, we obtain the following inequality.

Theorem 3. *Let $\tau \in \mathbb{R}$ be a nonzero constant. Then inequality*

$$e^{a+b} > \frac{b^\tau e^b - a^\tau e^a}{b^\tau - a^\tau} \quad (10)$$

for all $a > 0$ and $b > 0$ with $a \neq b$ holds if and only if $\tau \geq 1$ and reverses if and only if $\tau < 0$.

In particular, inequality

$$e^{a+b} > \frac{be^b - ae^a}{b - a} \quad (11)$$

is valid for all $a > 0$ and $b > 0$ with $a \neq b$, which is equivalent to the following integral inequality

$$e^{a+b} > \frac{1}{a-b} \int_b^a (1+u)e^u du. \quad (12)$$

2. PROOFS OF MAIN RESULTS

Proof of Theorem 1. Direct calculation arrives at

$$\begin{aligned} \Lambda_{p,q}(x) &= \int_0^\infty \frac{t}{t^2 + x^2} \left(\frac{1}{e^{2\pi pt} - 1} - \frac{q}{e^{2\pi t} - 1} \right) dt \\ &= \int_0^\infty \frac{1}{t^2 + x^2} \frac{t}{e^{2\pi t} - 1} \left(\frac{e^{2\pi t} - 1}{e^{2\pi pt} - 1} - q \right) dt \end{aligned} \quad (13)$$

and

$$\begin{aligned}\Phi_{p,q}(x) &= \int_0^\infty \frac{t}{t^2 + 4x^2} \left(\frac{1}{e^{\pi pt} + 1} - \frac{q}{e^{\pi t} + 1} \right) dt \\ &= \int_0^\infty \frac{1}{t^2 + 4x^2} \frac{t}{e^{\pi t} + 1} \left(\frac{e^{\pi t} + 1}{e^{\pi pt} + 1} - q \right) dt.\end{aligned}\quad (14)$$

Let

$$\omega_{r,s}(t) = \frac{e^{rt} - 1}{e^{st} - 1} \quad \text{and} \quad \chi_{r,s}(t) = \frac{e^{rt} + 1}{e^{st} + 1} \quad (15)$$

in $t \in (0, \infty)$ for positive real numbers $r > 0$ and $s > 0$. The L'Hôpital rule yields

$$\lim_{t \rightarrow 0^+} \omega_{r,s}(t) = \frac{r}{s}, \quad \lim_{t \rightarrow 0^+} \chi_{r,s}(t) = 1, \quad (16)$$

and

$$\lim_{t \rightarrow \infty} \omega_{r,s}(t) = \lim_{t \rightarrow \infty} \chi_{r,s}(t) = \begin{cases} 0, & r < s, \\ \infty, & r > s. \end{cases} \quad (17)$$

Direct differentiation and standard argument gives that

$$\frac{d\omega_{r,s}(t)}{dt} = \frac{(r-s)e^{(r+s)t} - (re^{rt} - se^{st})}{(e^{st} - 1)^2}$$

and

$$\frac{d\chi_{r,s}(t)}{dt} = \frac{(r-s)e^{(r+s)t} - (se^{st} - re^{rt})}{(e^{st} + 1)^2}.$$

The requirement $\frac{d\omega_{r,s}(t)}{dt} \leq 0$ is equivalent with

$$(r-s)e^{(r+s)t} \leq re^{rt} - se^{st},$$

$$(u-v)e^{u+v} \leq ue^u - ve^v,$$

$$ue^u(e^v - 1) \leq ve^v(e^u - 1),$$

$$\frac{ue^u}{e^u - 1} \leq \frac{ve^v}{e^v - 1},$$

where $u = rt > 0$ and $v = st > 0$. Since

$$\frac{d}{dx} \left(\frac{xe^x}{e^x - 1} \right) = \frac{e^x(e^x - x - 1)}{(e^x - 1)^2} > 0$$

for $x > 0$, the function $\frac{xe^x}{e^x - 1}$ is increasing in $x \in (0, \infty)$. This implies that $\frac{d\omega_{r,s}(t)}{dt} \leq$

0 in $(0, \infty)$ if and only if $r \leq s$, and then $\omega_{r,s}(t)$ is $\begin{cases} \text{decreasing} \\ \text{increasing} \end{cases}$ in $t \in (0, \infty)$ if

and only if $r \lesseqgtr s$. Therefore, when $p > 1$,

$$-q < \frac{e^{2\pi t} - 1}{e^{2\pi pt} - 1} - q < \frac{1}{p} - q;$$

when $0 < p < 1$,

$$\frac{1}{p} - q < \frac{e^{2\pi t} - 1}{e^{2\pi pt} - 1} - q < \infty.$$

Thus, if $p > 1$ and $q \leq 0$ or $0 < p < 1$ and $pq \leq 1$, the function $\Lambda_{p,q}(x)$ is positive and decreasing; if $p > 1$ and $pq \geq 1$, it is negative and increasing in $(0, \infty)$.

The requirement $\frac{d\chi_{r,s}(t)}{dt} \lesseqgtr 0$ is equivalent with

$$\begin{aligned} (r-s)e^{(r+s)t} &\lesseqgtr se^{st} - re^{rt}, \\ (u-v)e^{u+v} &\lesseqgtr ve^v - ue^u, \\ ue^u(e^v+1) &\lesseqgtr ve^v(e^u+1), \\ \frac{ue^u}{e^u+1} &\lesseqgtr \frac{ve^v}{e^v+1}, \end{aligned}$$

where $u = rt > 0$ and $v = st > 0$. Since

$$\frac{d}{dx} \left(\frac{xe^x}{e^x+1} \right) = \frac{e^x(e^x+x+1)}{(e^x+1)^2} > 0$$

for $x > 0$, the function $\frac{xe^x}{e^x+1}$ is increasing in $x \in (0, \infty)$. This implies that $\frac{d\chi_{r,s}(t)}{dt} \lesseqgtr$

0 in $(0, \infty)$ if and only if $r \lesseqgtr s$, and then $\chi_{r,s}(t)$ is $\begin{cases} \text{decreasing} \\ \text{increasing} \end{cases}$ in $t \in (0, \infty)$ if

and only if $r \lesseqgtr s$. Therefore, when $p > 1$,

$$-q < \frac{e^{\pi t} + 1}{e^{\pi pt} + 1} - q < 1 - q;$$

when $0 < p < 1$,

$$1 - q < \frac{e^{\pi t} + 1}{e^{\pi pt} + 1} - q < \infty.$$

Thus, if $p > 1$ and $q \leq 0$ or $0 < p < 1$ and $q \leq 1$, the function $\Phi_{p,q}(x)$ is positive and decreasing; if $p > 1$ and $q \geq 1$, it is negative and increasing in $(0, \infty)$. The proof of Theorem 1 is complete. \square

Proof of Theorem 2. Straightforward computation yields

$$\begin{aligned}\Lambda_{p,q}(x) &= \int_0^\infty \frac{t}{e^{2\pi t} - 1} \left(\frac{1}{t^2 + p^2 x^2} - \frac{q}{t^2 + x^2} \right) dt \\ &= \int_0^\infty \frac{t}{e^{2\pi t} - 1} \frac{(1-q)t^2 + (1-p^2q)x^2}{(t^2 + p^2 x^2)(t^2 + x^2)} dt \\ &\triangleq \int_0^\infty \frac{t}{e^{2\pi t} - 1} \rho_{p,q;t}(x) dt,\end{aligned}\tag{18}$$

$$\begin{aligned}\Phi_{p,q}(x) &= \int_0^\infty \frac{t}{e^{\pi t} + 1} \left(\frac{1}{t^2 + 4p^2 x^2} - \frac{q}{t^2 + 4x^2} \right) dt \\ &= \int_0^\infty \frac{t}{e^{\pi t} + 1} \frac{(1-q)t^2 + 4(1-p^2q)x^2}{(t^2 + p^2 x^2)(t^2 + 4x^2)} dt \\ &\triangleq \int_0^\infty \frac{t}{e^{\pi t} + 1} \varrho_{p,q;t}(x) dt.\end{aligned}\tag{19}$$

By standard argument, we have

$$\begin{aligned}\frac{d\rho_{p,q;t}(x)}{dx} &= \frac{2xt^4[p^2(p^2q-1)u^2 + 2p^2(q-1)u + (q-p^2)]}{(t^2 + p^2x^2)^2(t^2 + x^2)^2}, \\ \frac{d\varrho_{p,q;t}(x)}{dx} &= \frac{2xt^4[16p^2(p^2q-1)u^2 + 8p^2(q-1)u + (4q-p^2-3p^2q)]}{(t^2 + p^2x^2)^2(t^2 + 4x^2)^2},\end{aligned}$$

where $u = \left(\frac{x}{t}\right)^2 > 0$. Hence, if either

$$\begin{cases} p^2q - 1 \leq 0 \\ q \leq 0 \end{cases} \quad \text{or} \quad \begin{cases} p^2q - 1 = 0 \\ q - 1 \leq 0 \\ q - p^2 \leq 0, \end{cases}$$

then the derivative $\frac{d\rho_{p,q;t}(x)}{dx} \leq 0$; if either

$$\begin{cases} p^2q - 1 \leq 0 \\ q(p^2 - 1)[(1 + 3q)p^2 - 4] \leq 0 \end{cases} \quad \text{or} \quad \begin{cases} p^2q - 1 = 0 \\ q - 1 \leq 0 \\ 4q - p^2 - 3p^2q \leq 0, \end{cases}$$

then the derivative $\frac{d\varrho_{p,q;t}(x)}{dx} \leq 0$.

Consequently, if either $q \leq 0$ or $0 < q = \frac{1}{p^2} \leq 1$ and $p^4 \geq 1$ then $\frac{d\rho_{p,q;t}(x)}{dx} \leq 0$ and the function $\Lambda_{p,q}(x)$ is decreasing in $x \in (0, \infty)$; if $q = \frac{1}{p^2} \geq 1$ and $p^4 \leq 1$ then $\frac{d\rho_{p,q;t}(x)}{dx} \geq 0$ and the function $\Lambda_{p,q}(x)$ is increasing in $x \in (0, \infty)$; if either $p^2q < 1$ and $q(p^2 - 1)[(1 + 3q)p^2 - 4] \leq 0$ or $p^2q = 1$ and $0 < q \leq 1$ then $\frac{d\varrho_{p,q;t}(x)}{dx} \leq 0$ and the function $\Phi_{p,q}(x)$ is decreasing in $x \in (0, \infty)$; if either $p^2q > 1$ and

$(p^2 - 1)[(1 + 3q)p^2 - 4] \leq 0$ or $p^2q = 1$ and $q \geq 1$ then $\frac{d\varrho_{p,q;t}(x)}{dx} \geq 0$ and the function $\Phi_{p,q}(x)$ is increasing in $x \in (0, \infty)$. The proof of Theorem 2 is complete. \square

Proof of Theorem 3. Without loss of generality, assume $b > a > 0$ in (10). Then it can be rearranged as

$$\frac{b^\tau e^b}{e^b - 1} > \frac{a^\tau e^a}{e^a - 1}.$$

Direct calculation gives

$$\frac{d}{dx} \left(\frac{x^\tau e^x}{e^x - 1} \right) = \frac{x^{\tau-1} e^x \left(\tau - \frac{x}{e^x - 1} \right)}{e^x - 1}.$$

It is easy to see that the function $\frac{x}{e^x - 1}$ is decreasing in $(0, \infty)$, with

$$\lim_{x \rightarrow 0^+} \frac{x}{e^x - 1} = 1 \quad \text{and} \quad \lim_{x \rightarrow \infty} \frac{x}{e^x - 1} = 0.$$

Hence, the function $\frac{x^\tau e^x}{e^x - 1}$ is increasing (or decreasing, respectively) in $x \in (0, \infty)$ if and only if $\tau \geq 1$ (or $\tau < 0$, respectively). Inequality (10) follows. \square

REFERENCES

- [1] Ch.-P. Chen and F. Qi, *Completely monotonic functions related to the gamma functions*, RGMIA Res. Rep. Coll. **8** (2005), no. 2, Art. 3. Available online at <http://rgmia.vu.edu.au/v8n2.html>.
- [2] Ch.-P. Chen and F. Qi, *Logarithmically completely monotonic functions related to gamma function and remainder of Binet's formula with an application*, submitted.
- [3] I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series, and Products*, Sixth Edition, Academic Press, New York, 2000.
- [4] S. Guo, *Complete monotonicity of difference of remainder of Binet's formula*, private communication.
- [5] J.-Ch. Kuang, *Chángyòng Bùděng Shì (Applied Inequalities)*, 3rd ed., Shandong Science and Technology Press, Ji'nan City, Shandong Province, China, 2004. (Chinese)
- [6] W. Magnus, F. Oberhettinger and R. P. Soni, *Formulas and Theorems for the Special Functions of Mathematical Physics*, Springer, Berlin-New York, 1966.
- [7] F. Qi *Certain logarithmically N -alternating monotonic functions involving gamma and q -gamma functions*, RGMIA Res. Rep. Coll. **8** (2005), no. 3, Art. 5. Available online at <http://rgmia.vu.edu.au/v8n3.html>.
- [8] F. Qi, B.-N. Guo, and Ch.-P. Chen, *Some completely monotonic functions involving the gamma and polygamma functions*, J. Austral. Math. Soc. **79** (2006), no. 1, in press. RGMIA Res. Rep. Coll. **7** (2004), no. 1, Art. 5, 31–36. Available online at <http://rgmia.vu.edu.au/v7n1.html>.

- [9] Zh.-X. Wang and D.-R. Guo, *Special Functions*, Translated from the Chinese by D.-R. Guo and X.-J. Xia, World Scientific Publishing, Singapore, 1989.
- [10] Zh.-X. Wang and D.-R. Guo, *Tèshū Hánshù Gàilùn*, The Series of Advanced Physics of Peking University, Peking University Press, Beijing, China, 2000. (Chinese)
- [11] D. V. Widder, *The Laplace Transform*, Princeton Univ. Press, Princeton, New Jersey, 1941.

(F. Qi) RESEARCH INSTITUTE OF MATHEMATICAL INEQUALITY THEORY, HENAN POLYTECHNIC UNIVERSITY, JIAOZUO CITY, HENAN PROVINCE, 454010, CHINA

E-mail address: `qifeng@hpu.edu.cn`, `fengqi618@member.ams.org`, `qifeng618@hotmail.com`, `qifeng618@msn.com`

URL: `http://rgmia.vu.edu.au/qi.html`

(D.-W. Niu) SCHOOL OF MATHEMATICS AND INFORMATICS, HENAN POLYTECHNIC UNIVERSITY, JIAOZUO CITY, HENAN PROVINCE, 454010, CHINA

E-mail address: `nnddw@tom.com`

(B.-N. Guo) SCHOOL OF MATHEMATICS AND INFORMATICS, HENAN POLYTECHNIC UNIVERSITY, JIAOZUO CITY, HENAN PROVINCE, 454010, CHINA

E-mail address: `guobaini@hpu.edu.cn`