ON COMPARISON OF TWO-PARAMETER HOMOGENEOUS SYMMETRIC FUNCTIONS

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Abstract. A function may be decomposed into the product of a positive function and another monotone one. Proceeding from the train of thoughts, Pales's comparison theorem is improved, a new proof of which is given. As a class of special functions, the comparison problem for two-parameter functions are solved. Lastly, the comparison theorems of certain familiar two-parameter homogeneous functions are presented.

1. Introduction

The comparison theorem for Stolarsky means defined by

$$E(p,q;a,b)=\begin{cases}\left(\dfrac{q}{p}\dfrac{a^p-b^p}{a^q-b^q}\right)^{\frac{1}{p-q}} & p\neq q, pq\neq 0, a\neq b\\ L^{\frac{1}{p}}(a^p,b^p) & p\neq 0, q=0, a\neq b\\ L^{\frac{1}{q}}(a^q,b^q) & p=0, q\neq 0, a\neq b\\ E^{\frac{1}{p}}(a^p,b^p) & p=q\neq 0, a\neq b\\ G(a,b) & p=q=0, a\neq b\\ a & a=b\end{cases}, \tag{1.1}$$

where $L(a,b)=(a-b)/\ln(a/b), E(a,b)=e^{-1}(a^a/b^b)^{\frac{1}{a-b}}, G(a,b)=\sqrt{ab}$ proved first by Leach and Scholander in [6] states that $E(p,q;a,b)\leq E(r,s;a,b)$ holds for all $a,b>0$ if and only if

$$p+q\leq r+s \tag{1.2}$$

and

$$\begin{cases}(i)\ l(p,q)\leq l(r,s) & \text{if } \min(p,q,r,s)\geq 0 \text{ or } \max(p,q,r,s)\leq 0,\\ (ii)\ \mu(p,q)\leq\mu(r,s) & \text{if } \min m(p,q,r,s)<0<\max(p,q,r,s),\end{cases} \tag{1.3}$$

where

$$l(u,v)=\frac{u-v}{\ln(u/v)}, \mu(u,v):=\begin{cases}\dfrac{|u|-|v|}{u-v} & \text{if } u\neq v,\\ \operatorname{sgn}(u) & \text{if } u=v.\end{cases} \tag{1.4}$$

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In [7] Páles gave a proof of the comparison theorem for Gini means

$$G(p,q;a,b)=\begin{cases}\left(\dfrac{a^p+b^p}{a^q+b^q}\right)^{\frac{1}{p-q}} & p\neq q\\ Z^{\frac{1}{p}}(a^p,b^p) & p=q\neq 0\\ G(a,b) & p=q=0\end{cases}, \tag{1.5}$$

where $Z(a,b)=a^{\frac{a}{a+b}}b^{\frac{b}{a+b}}$, which states that $G(p,q,a,b)\leq G(r,s,a,b)$ holds for all $a,b>0$ if and only if (1.2) holds and

$$\begin{cases}(i)\ \min(p,q)\leq\min(r,s) & \text{if } \min(p,q,r,s)\geq 0,\\ (ii)\ \max(p,q)\leq\max(r,s) & \text{if } \max(p,q,r,s)\leq 0,\\ (iii)\ \mu(p,q)\leq\mu(r,s) & \text{if } \min m(p,q,r,s)<0<\max(p,q,r,s),\end{cases} \tag{1.6}$$

where $\mu(u,v)$ is defined by (1.4).

Using the same method, a new proof for the result of Leach and Sholander was presented in [8].

In addition, there are a number of papers concerning the comparison of Stolarsky and Gini means. (See, for instance [2], [3], [4], [5], [6], [9].).

In 1992 Páles [10] offered a unified treatment for these comparison problems on bounded intervals of the positive real, which is read as follows.

Theorem 1. [10, Theorem 2]*Let $\phi:\mathbb{R}\to\mathbb{R}$ be a four times differentiable even function such that $\phi''(0)>0$ and*

$$\phi''(x)>0, \phi'''(x)<0, \text{ and } (x\phi'''(x)/\phi''(x))'<0 \tag{1.7}$$

holds for all $x>0$. Define $\Phi_{p,q}(t)$ by

$$\Phi_{p,q}(t):=\begin{cases}\dfrac{\phi(pt)-\phi(qt)}{p-q} & \text{if } p\neq q,\\ t\phi'(pt), & \text{if } p=q.\end{cases} \tag{1.8}$$

Let p,q,r,s and $c>0$ be fixed real values. Then the inequality

$$\Phi_{p,q}(t)\leq\Phi_{r,s}(t) \tag{1.9}$$

holds for all $t\in[-c,c]$ if and only if

$$p+q\leq r+s \text{ and } \Phi_{p,q}(c)\leq\Phi_{r,s}(c). \tag{1.10}$$

Theorem 2. [10, Theorem D]*Let p,q,r,s be real parameters and $0<\alpha<\beta<\infty$. Then*

$$E_{p,q}(x,y)\leq E_{r,s}(x,y)$$

holds for all $x,y\in[\alpha,\beta]$ if and only if

$$p+q\leq r+s \text{ and } E_{p,q}(\alpha,\beta)\leq E_{r,s}(\alpha,\beta). \tag{1.11}$$

Theorem 3. [10, Theorem S]*Let p, q, r, s be real parameters and $0 < \alpha < \beta < \infty$. Then*

$$G_{p,q}(x, y) \le G_{r,s}(x, y)$$

holds for all $x, y \in [\alpha, \beta]$ if and only if

$$p + q \le r + s \text{ and } G_{p,q}(\alpha, \beta) \le G_{r,s}(\alpha, \beta). \tag{1.12}$$

Applying Theorem 1, a more generalized comparison problem for four-parameter mean containing the Stolarsky and Gini mean has been solved by Alfred Witkowski recently (see [11]).

On the other hand, the extended mean and Gini mean can be generalized to two-parameter functions by the author (see [12, 13, 14]). Naturally we ask if there are similar results for comparison of two-parameter functions. The purpose of the paper is to study the comparison problems. In section 2, the conditions and conclusion of Theorem 1 are refined, of which method of proof is improved. In section 3, the comparison problem of two-parameter homogeneous symmetric functions is solved. In section 4, some comparison theorems for some special two-parameter homogeneous functions are given.

2. Refinements of the Páles's Comparison Theorem

Theorem 4 (Refinement of Theorem 1)**.** *Let $\phi : \mathbb{R} \to \mathbb{R}$ be a four times differentiable even function such that $\phi''(x) > (<)0$ and $\operatorname{sgn}(x\phi'''(x)/\phi''(x))'$ is invariant for $x > 0$. Then the inequality $\Phi_{p,q}(t) \le \Phi_{r,s}(t)$ holds for all $t \in [-c, c]$ if and only if*

$$p + q \le (\ge) r + s \text{ and } \Phi_{p,q}(c) \le \Phi_{r,s}(c),$$

where $\Phi_{p,q}(t)$ is defined by (1.8).

The train of thoughts for proving comparison theorem is to decompose one function into the product of one positive function and another monotone function. For the latter there is the following obvious fact.

Lemma 1. *Suppose $f(x)$ is continuous and monotone on $[a, b]$, then $f(x) \ge (\le)0$ for all $x \in [a, b]$ if and only if $f(a) \ge (\le)0$ and $f(b) \ge (\le)0$.*

The following several Lemma will play an important role in the proof of Theorem 4.

Lemma 2. *Let $x_1, x_2, x_3 \in [a, b]$ and pairwise unequal. Define that*

$$[x_1, x_2; \phi] := \frac{\phi(x_1) - \phi(x_2)}{x_1 - x_2}, \tag{2.1}$$

$$[x_1, x_2, x_3; \phi] := \frac{[x_1, x_2; \phi] - [x_2, x_3; \phi]}{x_1 - x_3}. \tag{2.2}$$

Then we have

1) $[x_1, x_2, x_3; \phi]$ are symmetric with respect x_1, x_2 and x_3, i.e.

$$\begin{aligned}[x_1, x_2, x_3; \phi] &= [x_1, x_3, x_2; \phi] = [x_2, x_3, x_1; \phi] \\ &= [x_2, x_1, x_3; \phi] = [x_3, x_1, x_2; \phi] = [x_3, x_2, x_1; \phi].\end{aligned}$$

2) $[x_1,x_2,x_3;\phi]\geq(\leq)0$ *if and only if* $\phi(x)$ *is convex (concave) on* $[a,b]$*.*

3) (Mean Value Theorem) If f *is two times differentiable on* $[a,b]$ *and* $x_1,x_2,x_3\in[a,b]$*, then there is a* ξ *between the smallest and largest* x_i *(strictly between if the* x_i *are not all the same) such that*

$$[x_1,x_2,x_3;\phi]=\frac{\phi''(\xi)}{2!}. \tag{2.3}$$

Lemma 3. *Define*

$$V(t):=\frac{\Phi_{r,s}(t)-\Phi_{r,p}(t)}{s-p}\Big/\frac{\Phi_{p,q}(t)-\Phi_{p,r}(t)}{q-r}, \tag{2.4}$$

where $\Phi_{p,q}(t)$ *is defined by (1.8),* p,q,r,s *are pairwise unequal. If* $\phi''(x)>0(<0)$ *and* $\operatorname{sgn}(x\phi'''(x)/\phi''(x))'$ *is invariant for* $x>0$*, then* $V(t)$ *is strictly monotone in* t *on* $(0,\infty)$*; for* $t\in(-\infty,0)$ *the monotonicity is reversed.*

Proof. We first consider the case of $t>0$.

By Lemma 2, we get

$$\begin{aligned} V(t) &= \frac{[rt,st,pt;\phi]}{[pt,qt,rt;\phi]}=\frac{\frac{1}{st-rt}[st,pt;\phi]-[rt,pt;\phi]}{\frac{1}{qt-rt}[qt,pt;\phi]-[rt,pt;\phi]} \\ &= \frac{\frac{1}{s-r}\left[\frac{\phi(st)-\phi(pt)}{s-p}-\frac{\phi(rt)-\phi(pt)}{r-p}\right]}{\frac{1}{q-r}\left[\frac{\phi(qt)-\phi(pt)}{q-p}-\frac{\phi(rt)-\phi(pt)}{r-p}\right]}. \end{aligned}$$

By Mean Value Theorem there exist $m_{s,p}$ between s and p such that

$$t\phi'(m_{s,p}t)=\frac{\phi(st)-\phi(pt)}{s-p}. \tag{2.5}$$

Thus $V(t)$ can be rewritten as

$$V(t)=\frac{(s-r)^{-1}\left[\phi'(m_{s,p}t)-\phi'(m_{r,p}t)\right]}{(q-r)^{-1}\left[\phi'(m_{q,p}t)-\phi'(m_{r,p}t)\right]}.$$

Using logarithmic derivation method, we have

$$\begin{aligned} \frac{V'(t)}{V(t)} &= \frac{m_{s,p}\phi''(m_{s,p}t)-m_{r,p}\phi''(m_{r,p}t)}{\phi'(m_{s,p}t)-\phi'(m_{r,p}t)}-\frac{m_{q,p}\phi''(m_{q,p}t)-m_{r,p}\phi''(m_{r,p}t)}{\phi'(m_{q,p}t)-\phi'(m_{r,p}t)} \\ &= \frac{1}{t}\left[\frac{g(m_{s,p}t)-g(m_{r,p}t)}{h(m_{s,p}t)-h(m_{r,p}t)}-\frac{g(m_{q,p}t)-g(m_{r,p}t)}{h(m_{q,p}t)-h(m_{r,p}t)}\right], \end{aligned} \tag{2.6}$$

where

$$g(x)=x\phi''(x),h(x)=\phi'(x). \tag{2.7}$$

Since $\phi(x)$ is an even function, then $g(x)$ and $h(x)$ are both odd functions, and then

$$g(x)=-g(|x|),h(x)=-h(|x|); \tag{2.8}$$

meanwhile from $\phi''(x) > 0$ we see that $h(x) = \phi'(x)$ is reversible. Therefore $V'(t)/V(t)$ can be rewritten further as

$$\begin{aligned}
\frac{V'(t)}{V(t)} &= \frac{1}{t}\left[\frac{g(|m_{s,p}|t) - g(|m_{r,p}|t)}{h(|m_{s,p}|t) - h(|m_{r,p}|t)} - \frac{g(|m_{q,p}|t) - g(|m_{r,p}|t)}{h(|m_{q,p}|t) - h(|m_{r,p}|t)}\right] \\
&= \frac{h(|m_{s,p}|t) - h(|m_{q,p}|t)}{t} \frac{\frac{g(|m_{s,p}|t)-g(|m_{r,p}|t)}{h(|m_{s,p}|t)-h(|m_{r,p}|t)} - \frac{g(|m_{q,p}|t)-g(|m_{r,p}|t)}{h(|m_{q,p}|t)-h(|m_{r,p}|t)}}{h(|m_{s,p}|t)-h(|m_{q,p}|t)} \\
&= \frac{h(|m_{s,p}|t) - h(|m_{q,p}|t)}{t} \cdot [h(|m_{s,p}|t), h(|m_{r,p}|t), h(|m_{q,p}|t); g \circ h^{-1}] \\
&= V_1(t) \cdot V_2(t). \qquad (2.9)
\end{aligned}$$

From (2.8) and (2.5) we have

$$\begin{aligned}
V_1(t) &= \frac{h(|m_{s,p}|t) - h(|m_{q,p}|t)}{t} \\
&= \frac{-h(m_{s,p}t) + h(m_{q,p}t)}{t} \\
&= \frac{1}{t}[\phi'(m_{q,p}t) - \phi'(m_{s,p}t)] \\
&= \frac{1}{t}\left[\frac{\phi(qt) - \phi(pt)}{qt - tp} - \frac{\phi(st) - \phi(pt)}{st - pt}\right] \\
&= (q - s)[pt, qt, st; \phi]. \qquad (2.10)
\end{aligned}$$

On the other hand, By 3) of Lemma 2, there is a ξ between the smallest and largest among the $h(|m_{s,p}|t), h(|m_{r,p}|t)$ and $h(|m_{q,p}|t)$ such that
(2.11)

$$V_2(t) = [h(|m_{s,p}|t), h(|m_{r,p}|t), h(|m_{q,p}|t); g \circ h^{-1}] = \frac{1}{2}(g(h^{-1}(y)))''|_{y=\xi}.$$

From (2.7) by a calculation we get

$$(g(h^{-1}(y)))'' = \left(\frac{g'(x)}{h'(x)}\right)' / h'(x) = \left(\frac{x\phi'''(x)}{\phi''(x)}\right)' / \phi''(x) = \frac{\mathcal{K}(x)}{\phi''(x)}, \qquad (2.12)$$

where $x = h^{-1}(y)$. Since $h(x) = \phi'(x)$ is reversible, we understand that $x = h^{-1}(y)$ implies x between the smallest and largest $|m_{s,p}|t, |m_{r,p}|t$ and $|m_{q,p}|t$, namely, there is a $m > 0$ between the smallest and largest $|m_{s,p}|, |m_{r,p}|$ and $|m_{q,p}|$ such that $x = mt$. Consequently, we have

$$V_2(t) = (g(h^{-1}(y)))''|_{y=\xi} = \frac{\mathcal{K}(mt)}{\phi''(mt)}, \text{where } m > 0. \qquad (2.13)$$

From (2.9), (2.10) and (2.13), we obtain

$$\frac{V'(t)}{V(t)} = (q - s)[pt, qt, st; \phi]\frac{\mathcal{K}(mt)}{\phi''(mt)}. \qquad (2.14)$$

Moreover, in view of $\phi''(x) > (<)0$, there is

$$V(t) = [rt, st, pt; \phi]/[pt, qt, rt; \phi] > 0. \qquad (2.15)$$

It follows from (2.14) and (2.15) that

$$\begin{aligned} \operatorname{sgn} V'(t) &= \operatorname{sgn}(q-s)[pt,qt,st;\phi] \operatorname{sgn} \frac{\mathcal{K}(mt)}{\phi''(mt)} \\ &= \operatorname{sgn}(q-s) \operatorname{sgn} \mathcal{K}(mt). \end{aligned} \tag{2.16}$$

which indicates that $V(t)$ is monotone in t on $(0,\infty)$.

Next then let us consider another case of $t<0$.

Note $\mathcal{K}(x) = \big(x\phi'''(x)/\phi''(x)\big)'$ is an odd function, we have

$$\operatorname{sgn} V'(-t) = \operatorname{sgn}(q-s) \operatorname{sgn} \mathcal{K}(-mt) = -\operatorname{sgn}(q-s) \operatorname{sgn} \mathcal{K}(mt), \tag{2.17}$$

which shows that the monotonicity of $V(t)$ in t on $(-\infty,0)$ is just contrary to on $(0,\infty)$.

This Lemma is proved. ∎

Proof of Theorem 4. Without loss of generality, we assume $\phi''(x)>0$. Denote $\Delta(t) = \Phi_{p,q}(t) - \Phi_{r,s}(t)$.

1) If p,q,r,s are pairwise unequal, then

$$\begin{aligned} \Delta(t) &= \Phi_{p,q}(t) - \Phi_{r,s}(t) = (\Phi_{p,q}(t) - \Phi_{p,r}(t)) - (\Phi_{r,s}(t) - \Phi_{r,p}(t)) \\ &= \frac{\Phi_{p,q}(t) - \Phi_{p,r}(t)}{q-r} \left[q-r-(s-p)\frac{\dfrac{\Phi_{r,s}(t) - \Phi_{r,p}(t)}{s-p}}{\dfrac{\Phi_{p,q}(t) - \Phi_{p,r}(t)}{q-r}} \right] \\ &= t^2[pt,qt,rt;\phi]U(t), \end{aligned} \tag{2.18}$$

where

$$U(t) = (q-r) - (s-p)V(t), \tag{2.19}$$

$V(t)$ is defined by (2.4).

It is easy to verify that $V(t)$ is an even function and so is $U(t)$. By (2.14) we have

$$U'(t) = (s-p)V'(t) = (s-p)V(t)(q-s)[pt,qt,st;\phi]\frac{\mathcal{K}(mt)}{\phi''(mt)}, \tag{2.20}$$

it follows that

$$\operatorname{sgn} U'(t) = \operatorname{sgn}(s-p)\operatorname{sgn}(q-s)\operatorname{sgn}\mathcal{K}(mt) \text{ for } t>0, \tag{2.21}$$

where $\mathcal{K}(x) = (x\phi'''(x)/\phi''(x))', x>0$. This leads to $U(t)$ is strictly monotone on $\mathbb{R}_+$.

On the other hand, By (2.18) $U(t)$ can be expressed in the following form as

$$U(t) = \frac{\Delta(t)}{t^2[pt,qt,rt;\phi]} = \frac{\Phi_{p,q}(t) - \Phi_{r,s}(t)}{t^2[pt,qt,rt;\phi]} \tag{2.22}$$

According to Lemma 1 and (2.22), if $\phi''(x)>0$ then $\Delta(t) \le 0$ $t \in [-c,c]$ if and only if $U(t) \le (\ge) 0, t \in [-c,c]$ if and only if $U(0) \le (\ge) 0$ and

$U(c) \leq (\geq) 0$. But

$$\begin{aligned}
V(0) \quad : \quad &= \lim_{t\to 0} V(t) = \lim_{t\to 0} \frac{\dfrac{\Phi_{r,s}(t)-\Phi_{r,p}(t)}{s-p}}{\dfrac{\Phi_{p,q}(t)-\Phi_{p,r}(t)}{q-r}} = \lim_{t\to 0} \frac{[rt,st,pt;\phi]}{[pt,qt,rt;\phi]} \\
&= \lim_{t\to 0} \frac{\frac{1}{r-p}\left[\dfrac{\phi(rt)-\phi(st)}{r-s} - \dfrac{\phi(pt)-\phi(st)}{p-s}\right]}{\frac{1}{r-p}\left[\dfrac{\phi(rt)-\phi(qt)}{r-q} - \dfrac{\phi(pt)-\phi(qt)}{p-q}\right]} \\
&= \lim_{t\to 0} \frac{\dfrac{r^2\phi''(st)-s^2\phi''(rt)}{r-s} - \dfrac{p^2\phi''(st)-s^2\phi''(pt)}{p-s}}{\dfrac{r^2\phi''(rt)-q^2\phi''(qt)}{r-q} - \dfrac{p^2\phi''(pt)-q^2\phi''(qt)}{p-q}} \\
&= \frac{\dfrac{r^2\phi''(0)-s^2\phi''(0)}{r-s} - \dfrac{p^2\phi''(0)-s^2\phi''(0)}{p-s}}{\dfrac{r^2\phi''(0)-q^2\phi''(0)}{r-q} - \dfrac{p^2\phi''(0)-q^2\phi''(0)}{p-q}} = 1,
\end{aligned}$$

so

$$U(0) := \lim_{t\to 0} U(t) = \lim_{t\to 0} [(q-r)-(s-p)V(t)] = p+q-r-s;$$

that $U(c) \leq (\geq) 0$ i.e. $\Delta(t) = \Phi_{p,q}(c) - \Phi_{r,s}(c) = c^2[pc,qc,rc;\phi]U(c) \leq 0$.

Thus it can be seen that $\Phi_{p,q}(t) \leq \Phi_{r,s}(t)$ holds for all $t \in [-c,c]$ if and only if $p+q \leq (\geq) r+s$ and $\Phi_{p,q}(c) \leq \Phi_{r,s}(c)$.

2) If some of p,q,r,s are equal, for instance $p=r$, then we can take their limits at $p=r$ in processes of proof above. It is clear that our desired result still holds, of which processes in detail are omitted. ∎

Theorem 5. *Let $\phi : \mathbb{R} \to \mathbb{R}$ be a four times differentiable even function such that $\phi''(x) > (<) 0$ and $\operatorname{sgn}(x\phi'''(x)/\phi''(x))'$ is invariant. Then the inequality $\Phi_{p,q}(t) \leq \Phi_{r,s}(t)$ holds for all $t \in \mathbb{R}$ if and only if $p+q \leq (\geq) r+s$ and other conditions satisfying that*

$$\lim_{t\to\infty} \frac{\Phi_{p,q}(t)-\Phi_{r,s}(t)}{\Psi(p,q;r,s,t)} \leq 0, \tag{2.23}$$

where $\Phi_{p,q}(t)$ is defined by (1.8), $\Psi(p,q;r,s,t) > 0$.

Proof. We prove only in the case p,q,r,s are pairwise unequal. The proof in other cases is similar to part two of Theorem 4.

By (2.18) and $\phi''(x) > (<) 0$ that $\Delta(t) = \Phi_{p,q}(t) - \Phi_{r,s}(t) \leq 0$ for all $t \in R$ if and only if $U(t) \leq (\geq) 0$ for all $t \in R$. From process of proof of Theorem 4 and (2.18) we see that $U(t)$ is even and monotone on R. By Lemma 1 $U(t) \leq (\geq) 0$ for all $t \in R$ if and only if $U(0) \leq (\geq) 0$ and $U(\infty) \leq (\geq) 0$.

By $U(0) \leq (\geq) 0$ we can deduce that $p+q \leq (\geq) r+s$ holds. From $U(\infty) \leq (\geq) 0$ and (2.22) we can obtain that

$$\lim_{t\to\infty} \frac{\Phi_{p,q}(t)-\Phi_{r,s}(t)}{|t^2[pt,qt,rt;\phi]|} \leq 0,$$

setting $\Psi(p,q;r,s,t) = |t^2[pt,qt,rt;\phi]|$, we get (2.23).

Thus we end the proof. ∎

From the processes of proof of Theorem 4, we easily obtain the theorem, of which processes of proof are omitted.

Theorem 6. *Let $\phi : \mathbb{R}_+ \to \mathbb{R}$ be a four times differentiable function such that $\phi''(x) > (<) 0$, $\operatorname{sgn}(x\phi'''(x)/\phi''(x))'$ is invariant for $x > 0$ and $\Phi_{p,q}(+0) \triangleq \lim_{t\to 0, t>0} \Phi_{p,q}(t)$ exists with $p, q > 0$. Then the inequality $\Phi_{p,q}(t) \leq \Phi_{r,s}(t)$ holds for all $t \in (0, c]$ and $p, q, r, s > 0$ if and only if*

$$\Phi_{p,q}(+0) \leq \Phi_{r,s}(+0) \text{ and } \Phi_{p,q}(c) \leq \Phi_{r,s}(c).$$

3. Comparison of Two-parameter Homogeneous Symmetric Functions

For the sake of simplicity, we introduce the concept and notations of two-parameter homogeneous functions.

Definition 1. *Assume $f\colon \mathbb{U}(\subseteqq \mathbb{R}_+ \times \mathbb{R}_+) \to \mathbb{R}_+$ is a homogeneous function for variable x and y, and is continuous and first-time partial derivatives exist, $(a,b) \in \mathbb{R}_+ \times \mathbb{R}_+$ with $a \neq b$, $(p,q) \in \mathbb{R} \times \mathbb{R}$. If $(1,1) \notin U$, and then define that*

$$\mathcal{H}_f(a,b;p,q) = \left[\frac{f(a^p,b^p)}{f(a^q,b^q)}\right]^{\frac{1}{p-q}} (p \neq q, pq \neq 0), \tag{3.1}$$

$$\mathcal{H}_f(a,b;p,p) = \lim_{q\to p} \mathcal{H}_f(a,b;p,q) = G_{f,p}(a,b)(p = q \neq 0). \tag{3.2}$$

where $G_{f,p}(a,b) = G_f^{\frac{1}{p}}(a^p,b^p)$,

$$G_f(x,y) = \exp\left[\frac{xf_x(x,y)\ln x + yf_y(x,y)\ln y}{f(x,y)}\right], \tag{3.3}$$

in which $f_x(x,y)$ and $f_y(x,y)$ denote first order partial derivative to first and second variable of $f(x,y)$, respectively.

If $(1,1) \in U$, then define further

$$\mathcal{H}_f(a,b;p,0) = \left[\frac{f(a^p,b^p)}{f(1,1)}\right]^{\frac{1}{p}} (p \neq 0, q = 0), \tag{3.4}$$

$$\mathcal{H}_f(a,b;0,q) = \left[\frac{f(a^q,b^q)}{f(1,1)}\right]^{\frac{1}{q}} (p = 0, q \neq 0), \tag{3.5}$$

$$\mathcal{H}_f(a,b;0,0) = \lim_{p\to 0} \mathcal{H}_f(a,b;p,0) = a^{\frac{f_x(1,1)}{f(1,1)}} b^{\frac{f_y(1,1)}{f(1,1)}} (p = q = 0). \tag{3.6}$$

Concerning the two-parameter homogeneous functions, there are some useful and interesting results on its monotonicity and log-convexity, in which the following two decision functions play a important role (see [12, 13, 14]) that are:

$$I = I(x,y) = \frac{\partial^2 \ln f(x,y)}{\partial x \partial y} = (\ln f(x,y))_{xy} = (\ln f)_{xy}, \tag{3.7}$$

$$J = J(x,y) = (x-y)\frac{\partial (xI)}{x} = (x-y)(xI)_x. \tag{3.8}$$

In what follows we will encounter another key decision function:

$$K = K(x,y) = (x-y)\frac{\partial (J(x,y)/(L(x,y)I(x,y)))}{\partial x} = (x-y)(J/(LI))_x, \tag{3.9}$$

where I and J are defined by (3.7) and (3.8), $L = L(x,y) = (x-y)/\ln(x/y)$ $(x, y > 0)$.

Next applying Theorem 4 to homogeneous functions, we have the following

Theorem 7. *Assume $f\colon \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ is an n-order homogeneous, symmetric and four times differentiable function satisfying that $I < (>) 0$ and* $\operatorname{sgn} K$ *is invariant, $(a,b) \in \mathbb{R}_+ \times \mathbb{R}_+$, $(p,q) \in \mathbb{R} \times \mathbb{R}$. Then the inequality*

$$\mathcal{H}_f(p,q;a,b) \le \mathcal{H}_f(r,s;a,b) \tag{3.10}$$

holds for all $a, b \in [\alpha, \beta]$ if and only if

$$p + q \le (\ge) r + s \text{ and } \mathcal{H}_f(p,q;\alpha,\beta) \le \mathcal{H}_f(r,s;\alpha,\beta).$$

Proof. Note $f(x,y)$ is an n-order symmetric function, so

$$\begin{aligned} \ln f(a^p, b^p) &= \ln[(b^p)^n f((\tfrac{a}{b})^p, 1)] \\ &= np + \ln f(e^{pt}, 1) \\ &= np + \ln \left(e^{\frac{pt}{2}}\right)^n f(e^{\frac{pt}{2}}, e^{-\frac{pt}{2}}) \\ &= np + \frac{npt}{2} + \ln f(e^{\frac{pt}{2}}, e^{-\frac{pt}{2}}) \\ &= np + \frac{npt}{2} + \ln f(e^{\frac{pt}{2}}, e^{-\frac{pt}{2}}), \end{aligned} \tag{3.11}$$

where $t = \ln(a/b)$. Put

$$\phi(t) = \ln f(e^{\frac{t}{2}}, e^{-\frac{t}{2}}),$$

then $\phi(t)$ is an even function defined on $(-\infty, \infty)$ because $f(x,y)$ is symmetric with respect to x and y. Therefore the comparison inequality

$$\mathcal{H}_f(p,q;a,b) \le \mathcal{H}_f(r,s;a,b) \text{ for all } a, b \in [\alpha, \beta]$$

is equivalent to

$$\Phi_{p,q}(t) \le \Phi_{r,s}(t), \text{for all } t \in [-\ln(\beta/\alpha), \ln(\beta/\alpha)],$$

where $\Phi_{p,q}(t)$ is defined by (1.8).

To complete the proof of this Theorem, we have to verify the $\operatorname{sgn}\left(t\phi'''(t)/\phi''(t)\right)'$ and $\operatorname{sgn}\phi''(t)$.

Set

$$e^t = x, 1 = y,$$

then $\phi(t) = \frac{nt}{2} + \ln f(x,y)$. By derivation calculations, we have

$$\begin{aligned} \phi'(t) &= \frac{f_x(x,y)}{f(x,y)}\frac{dx}{dt} = x(\ln f)_x, \\ \phi''(t) &= (x(\ln f)_x)_x \frac{dx}{dt} = x\left(x(\ln f)_x\right)_x. \end{aligned}$$

It is easy to verify that $x(\ln f)_x = xf_x(x,y)/f(x,y)$ is a zero-order function, then

$$x\left(x(\ln f)_x\right)_x + y\left(x(\ln f)_x\right)_y = 0, \text{i.e. } x\left(x(\ln f)_x\right)_x = -xy(\ln f)_{xy}$$

so

$$\begin{aligned} \phi''(t) &= -xy(\ln f)_{xy} = -xyI, \\ \phi'''(t) &= (-xyI)_x\frac{dx}{dt} = -xy\left(xI\right)_x, \\ \frac{t\phi'''(t)}{\phi''(t)} &= \frac{(xI)_x \ln x}{I} = \frac{(x-y)\left(xI\right)_x}{L(x,y)I} = \frac{J}{LI}, \\ \left(\frac{t\phi'''(t)}{\phi''(t)}\right)' &= \left(\frac{J}{LI}\right)_x\frac{dx}{dt} = x\left(\frac{J}{LI}\right)_x = \frac{x}{x-y}K. \end{aligned} \tag{3.12}$$

which show that

$$\begin{aligned} \operatorname{sgn}\phi''(t) &= -\operatorname{sgn} I, \\ \operatorname{sgn}\left(t\phi'''(t)/\phi''(t)\right)' &= \operatorname{sgn}\frac{x}{x-y}\operatorname{sgn} K = \operatorname{sgn} K \text{ for } t > 0. \end{aligned}$$

Applying Theorem 4, we complete the proof. ∎

As direct result of Theorem 5, there is the following theorem.

Theorem 8. *Assume $f\colon \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ is an n-order homogeneous, symmetric and four times differentiable function satisfying conditions that $I < 0$ and $\operatorname{sgn} K$ is invariant, $(a,b) \in \mathbb{R}_+ \times \mathbb{R}_+, (p,q) \in \mathbb{R} \times \mathbb{R}$.*

1) If $f(+0,1) := \lim\limits_{x\to 0, x>0} f(x,1) = \gamma = 0$ but exist $\delta > 0$ such that $\lim\limits_{x\to\infty} x^\delta f(e^{-x},1) = \lambda > 0$, then the inequality $\mathcal{H}_f(p,q;a,b) \le \mathcal{H}_f(r,s;a,b)$ holds for all $(a,b) \in \mathbb{R}_+ \times \mathbb{R}_+$ if and only if both (1.2) and (1.3) hold. If $I > 0$, then inequalities (1.2) is reversed.

2) If $f(+0,1) := \lim\limits_{x\to 0, x>0} f(x,1) = \gamma \ne 0$, then the inequality $\mathcal{H}_f(p,q;a,b) \le \mathcal{H}_f(r,s;a,b)$ holds for all $(a,b) \in \mathbb{R}_+ \times \mathbb{R}_+$ if and only if both (1.2) and (1.6) hold. If $I > 0$, then (1.2) and (i), (ii) of (1.6) are reversed.

Proof. Since $f(x,y)$ is symmetric with respect to x and y, we have

$$\phi(pt) = \ln f(e^{\frac{pt}{2}}, e^{-\frac{pt}{2}}) = \ln f(e^{\frac{|pt|}{2}}, e^{-\frac{|pt|}{2}}) = \frac{|pt|}{2}n + \ln f(e^{-|pt|}, 1), \tag{3.13}$$

and then

$$\begin{aligned}\Phi_{p,q}(t) - \Phi_{r,s}(t) &= \frac{\phi(pt)-\phi(qt)}{p-q} - \frac{\phi(rt)-\phi(st)}{r-s} \\ &= \frac{1}{2}\left(\frac{|p|-|q|}{p-q} - \frac{|r|-|s|}{r-s}\right)|t| \\ &\quad + \frac{\ln f(e^{-|pt|},1)-\ln f(e^{-|qt|},1)}{p-q} - \frac{\ln f(e^{-|rt|},1)-\ln f(e^{-|st|},1)}{r-s}.\end{aligned}$$

Assume that p, q, r, s are pairwise unequal in what follows in proof of this Theorem. If $p = q$ or $r = s$, we can regard it as a limit of $p \to q$ or $r \to s$.

1)If $f(+0,1) := \lim\limits_{x\to 0, x>0} f(x,1) = \gamma = 0$ but exist $\delta > 0$ such that $\lim\limits_{x\to\infty} x^\delta f(e^{-x},1) = \lambda > 0$, then we have

$$\begin{aligned}&\frac{\ln f(e^{-|pt|},1)-\ln f(e^{-|qt|},1)}{p-q} - \frac{\ln f(e^{-|rt|},1)-\ln f(e^{-|st|},1)}{r-s} \\ =\ &\frac{\ln |pt|^\delta f(e^{-|pt|},1)-\ln |qt|^\delta f(e^{-|qt|},1)-\ln(|pt|^\delta/|qt|^\delta)}{p-q} \\ &-\frac{\ln |rt|^\delta f(e^{-|rt|},1)-\ln |st|^\delta f(e^{-|st|},1)-\ln(|rt|^\delta/|st|^\delta)}{r-s} \\ \to\ &-\delta\left(\frac{\ln(|p|/|q|)}{p-q} - \frac{\ln(|r|/|s|)}{r-s}\right) \quad (t\to\infty).\end{aligned}$$

(i) For $\min(p,q,r,s) < 0 < \max(p,q,r,s)$. We have

$$\begin{aligned}\lim_{t\to\infty}\frac{\Phi_{p,q}(t)-\Phi_{r,s}(t)}{t} &= \lim_{t\to\infty}\Big\{\frac{1}{2}\left(\frac{|p|-|q|}{p-q} - \frac{|r|-|s|}{r-s}\right)\frac{|t|}{t} \\ &\quad + \frac{1}{t}\left[\frac{\ln f(e^{-|pt|},1)-\ln f(e^{-|qt|},1)}{p-q} - \frac{\ln f(e^{-|rt|},1)-\ln f(e^{-|st|},1)}{r-s}\right]\Big\} \\ &= \frac{1}{2}\left(\frac{|p|-|q|}{p-q} - \frac{|r|-|s|}{r-s}\right).\end{aligned}$$

(ii) For $\min(p,q,r,s) > 0$ or $\max(p,q,r,s) > 0$. Note

$$\frac{|p|-|q|}{p-q} - \frac{|r|-|s|}{r-s} = 0$$

we have

$$\lim_{t\to\infty}[\Phi_{p,q}(t) - \Phi_{r,s}(t)] = -\delta\left(\frac{\ln(|p|/|q|)}{p-q} - \frac{\ln(|r|/|s|)}{r-s}\right). \tag{3.14}$$

2) If $f(+0,1) := \lim\limits_{x\to 0, x>0} f(x,1) = \gamma \neq 0$, then

(i) For $\min(p,q,r,s) < 0 < \max(p,q,r,s)$. Note

$$\lim_{t\to\infty}\left[\frac{\ln f(e^{-|pt|},1)-\ln f(e^{-|qt|},1)}{p-q} - \frac{\ln f(e^{-|rt|},1)-\ln f(e^{-|st|},1)}{r-s}\right] = 0,$$

so we have

$$\lim_{t\to\infty}\frac{\Phi_{p,q}(t)-\Phi_{r,s}(t)}{t}=\frac{1}{2}\left(\frac{|p|-|q|}{p-q}-\frac{|r|-|s|}{r-s}\right).$$

(ii) For $\min(p,q,r,s)>0$. Assume $p\geq q, r\geq s$, then $\min(p,q,r,s)=\min(q,s)$. If $q<s$, applying L'Hospital rule (two times) and by (3.12) we get

$$\begin{aligned}
&\lim_{t\to\infty}\frac{\Phi_{p,q}(t)-\Phi_{r,s}(t)}{e^{-t\min(q,s)}}\\
=&\lim_{t\to\infty}\left[\frac{\ln f(e^{-|pt|},1)-\ln f(e^{-|qt|},1)}{(p-q)e^{-qt}}-\frac{\ln f(e^{-|rt|},1)-\ln f(e^{-|st|},1)}{(r-s)e^{-qt}}\right]\\
=&\lim_{t\to\infty}\left[\frac{-e^{-pt}p^2I(e^{-pt},1)+e^{-qt}q^2I(e^{-qt},1)}{q^2(p-q)e^{-qt}}-\frac{-e^{-rt}r^2I(e^{-rt},1)+e^{-st}s^2I(e^{-st},1)}{q^2(r-s)e^{-qt}}\right]\\
=&\frac{I(0,1)}{p-q},
\end{aligned}$$

in this way, we can get

$$\begin{aligned}
\lim_{t\to\infty}\frac{\Phi_{p,q}(t)-\Phi_{r,s}(t)}{e^{-t\min(q,s)}}&=-\frac{I(0,1)}{r-s}\text{ if } q>s,\\
\lim_{t\to\infty}\frac{\Phi_{p,q}(t)-\Phi_{r,s}(t)}{e^{-t\min(q,s)}}&=\frac{I(0,1)}{p-q}-\frac{I(0,1)}{r-s}\text{ if } q=s.
\end{aligned}$$

Because $\frac{1}{p-q}-\frac{1}{r-s}=\frac{r-p}{(p-q)(r-s)}\geq 0$ under the condition $p+q\leq r+s$ and note $I(x,y)<(>)0$, we obtain

$$\lim_{t\to\infty}\frac{\Phi_{p,q}(t)-\Phi_{r,s}(t)}{e^{-t\min(q,s)}}=\begin{cases}\frac{I(0,1)}{p-q}<(>)0, & q<s;\\ \frac{I(0,1)}{p-q}-\frac{I(0,1)}{r-s}\geq(\leq)0, & q=s;\\ -\frac{I(0,1)}{r-s}>(<)0, & q>s.\end{cases}\tag{3.15}$$

(iii) For $\max(p,q,r,s)<0$. Similarly, under the same assumptions $p\geq q, r\geq s$, if $I(x,y)<(>)0$ then there are

$$\lim_{t\to\infty}\frac{\Phi_{p,q}(t)-\Phi_{r,s}(t)}{e^{t\max(p,r)}}=\begin{cases}-\frac{I(0,1)}{p-q}>(<)0, & p>r;\\ -\frac{I(0,1)}{p-q}+\frac{I(0,1)}{r-s}\leq(\geq)0, & p=r;\\ \frac{I(0,1)}{r-s}<(>)0, & p<r.\end{cases}$$

Since $p+q\leq(\geq)r+s$ is a common condition, using Corollary 5, we obtain the desired results.

The proof is completed. ∎

For homogeneous functions using Theorem 6 and 8, we will get

Theorem 9. *Assume $f\colon \mathbb{R}_+\times\mathbb{R}_+\to\mathbb{R}_+$ is an n-order homogeneous, symmetric and four times differentiable function satisfying the conditions that $I<(>)0$ and $\operatorname{sgn} K$ is invariant; $(a,b),(p,q),(r,s)\in\mathbb{R}_+\times\mathbb{R}_+$, $f(1,1)=0$ but $f_x(+1,1):=\lim_{x\to1,x>1}f_x(x,1)$ exists and is nonzero.*

1) If $f(+0,1):=\lim_{x\to0,x>0}f(x,1)=\gamma=0$ but exist $\delta>0$ such that $\lim_{x\to\infty}x^\delta f(e^{-x},1)=\lambda>0$, then the inequality $\mathcal{H}_f(p,q;a,b)\leq\mathcal{H}_f(r,s;a,b)$

holds for all $(a,b)\in\mathbb{R}_+\times\mathbb{R}_+$ *if and only if* $l(p,q)=l(r,s)$*, where* $l(u,v)=(u-v)/\ln(u/v)$*.*

2) If $f(+0,1):=\lim\limits_{x\to 0,x>0} f(x,1)=\gamma\neq 0$*, then the inequality* $\mathcal{H}_f(p,q;a,b)\le\mathcal{H}_f(r,s;a,b)$ *holds for all* $(a,b)\in\mathbb{R}_+\times\mathbb{R}_+$ *if and only if both inequalities*

$$l(p,q) \ \ge\ l(r,s), \tag{3.16}$$

$$\min(p,q) \ \le\ (\ge)\min(r,s). \tag{3.17}$$

hold.

Proof. By (3.13), for $p,q>0$ we have

$$\Phi_{p,q}(t)=\frac{\phi(pt)-\phi(qt)}{p-q}=\frac{\ln f(e^{-pt},1)-\ln f(e^{-qt},1)}{p-q},$$

letting t tend 0, since $f_x(1,1):=\lim\limits_{x\to 1} f_x(x,1)$ exists and is nonzero we get

$$\begin{aligned}\lim_{t\to 0}\Phi_{p,q}(t) &= \lim_{t\to 0}\frac{1}{p-q}\ln\frac{f(e^{-pt},1)}{f(e^{-qt},1)}\\ &= \lim_{t\to 0}\frac{1}{p-q}\ln\frac{-pe^{-pt}f_x(e^{-pt},1)}{-qe^{-qt}f_x(e^{-qt},1)}\\ &= \frac{\ln p-\ln q}{p-q}=\frac{1}{l(p,q)},\end{aligned}$$

likewise

$$\lim_{t\to 0}\Phi_{r,s}(t)=\frac{1}{l(r,s)},$$

i.e. $\Phi_{p,q}(0)$ and $\Phi_{r,s}(t)$ both exist. And then from $\Phi_{p,q}(0)\le\Phi_{r,s}(0)$ it follows that $\frac{1}{l(p,q)}\le\frac{1}{l(r,s)}$,i.e.

$$l(p,q)\ge l(r,s). \tag{3.18}$$

By Theorem 6, inequality (3.10) holds if and only if $\Phi_{p,q}(0)\le\Phi_{r,s}(0)$ and $\Phi_{p,q}(c)\le\Phi_{r,s}(c)$.

1) If $f(+0,1):=\lim\limits_{x\to 0,x>0} f(x,1)=\gamma=0$ but $\lim\limits_{x\to\infty} x^{\delta}f(e^{-x},1)=\lambda>0$, then for $p,q,r,s>0$ from (3.14) it follows that

$$\lim_{t\to\infty}\left[\Phi_{p,q}(t)-\Phi_{r,s}(t)\right]=-\delta\left(\tfrac{\ln(p/q)}{p-q}-\tfrac{\ln(r/s)}{r-s}\right)\le 0,$$

i.e.

$$l(p,q)\le l(r,s). \tag{3.19}$$

Combining (3.18) with (3.19), by Theorem 6 we can conclude that inequality $\mathcal{H}_f(p,q;a,b)\le\mathcal{H}_f(r,s;a,b)$ holds if and only if

$$l(p,q)=l(r,s),$$

which completes the proof of part one.

2) If $f(+0,1) := \lim_{x\to 0, x>0} f(x,1) = \gamma \neq 0$, then for $p,q,r,s>0$ from (3.15) it follows that

$$\min(p,q) \le (\ge) \min(r,s) \text{ if } I(0,1) < (>) 0. \tag{3.20}$$

Combining (3.18) with (3.20), we can deduce that by Theorem 6 inequality (3.10) holds if and only if both (3.16) and (3.17) hold.

Thus the proof is completed. ∎

4. Applications

As applications of theorems above, next let us observe comparison problems of certain familiar two-parameter homogeneous functions.

Example 1. *Let $f(x,y) = L(x,y) = \frac{x-y}{\ln x - \ln y}(x,y>0, x\neq y)$. Then $\mathcal{H}_L(p,q;a,b)$ is just the so-called extended mean (1.1). By some partial derivative calculations, we get*

$$I = (\ln f)_{xy} = \tfrac{xy\ln^2(x/y)-(x-y)^2}{xy(x-y)^2\ln^2(x/y)} = \tfrac{G^2(x,y)-L^2(x,y)}{xy(x-y)^2}.$$

From the well-known inequalities $L(x,y) > G(x,y)$ we have $I<0$. On the other hand,

$$\begin{aligned}
J &= (x-y)(xI)_x = \tfrac{2(x-y)^3-xy(x+y)\ln^3(x/y)}{xy(x-y)^2\ln^3(x/y)},\\
J/(LI) &= \tfrac{2(x-y)^3-xy(x+y)\ln^3(x/y)}{(x-y)[xy\ln^2(x/y)-(x-y)^2]} \overset{x/y=t}{=} \tfrac{2(t-1)^3-t(t+1)\ln^3 t}{t(t-1)\ln^2 t-(t-1)^3}\\
(J/(LI))_x &= \frac{1}{y}\left(\tfrac{2(t-1)^3-t(t+1)\ln^3 t}{t(t-1)\ln^2 t-(t-1)^3}\right)'\\
&= \frac{1}{y}\frac{2t^2\ln^5 t - t(t-1)(t+1)\ln^4 t-(t-1)^2(t^2+4t+1)\ln^3 t+5(t-1)^3(t+1)\ln^2 t-4(t-1)^4\ln t}{\left[t(t-1)\ln^2 t-(t-1)^3\right]^2}.
\end{aligned}$$

Set $(t-1)/\ln t = L$, then K can be expressed as

$$\begin{aligned}
K &= (x-y)\left(J/(LI)\right)_x\\
&= \tfrac{(x-y)\ln^5 t}{y}\tfrac{2t^2-t(t+1)L-(t^2+4t+1)L^2+5(t+1)L^3-4L^4}{\left[t(t-1)\ln^2 t-(t-1)^3\right]^2}\\
&= -\tfrac{(t-1)\ln^5 t\cdot\left[(L^2+L-2)t^2-(5L^3-4L^2-L)t+(4L^4-5L^3+L^2)\right]}{\left[t(t-1)\ln^2 t-(t-1)^3\right]^2}\\
&= -\tfrac{(t-1)\ln^5 t\cdot(L-1)(t-L)[(L+2)t-L(4L-1)]}{\left[t(t-1)\ln^2 t-(t-1)^3\right]^2}\\
&= \tfrac{(L-1)(t-L)(t-1)\ln^5 t}{y\left[t(t-1)\ln^2 t-(t-1)^3\right]^2}(2L^2-\frac{t+1}{2}L-t).
\end{aligned}$$

Obviously, $(t-1)\ln^5 t>0$; since $L=\frac{t-1}{\ln t}$ lies between 1 and t, then $(L-1)(t-L)>0$; put $\frac{t+1}{2}=A$, $\sqrt{x}=G$, then

$$(2L^2-\frac{t+1}{2}L-t) = (L+\tfrac{\sqrt{A^2+8G^2}-A}{4})(L-\tfrac{\sqrt{A^2+8G^2}+A}{4})$$

where $L+\frac{\sqrt{A^2+8G^2}-A}{4}>0$*, and* $L-\frac{\sqrt{A^2+8G^2}+A}{4}<0$ *follows from* $L<\frac{2A+G}{3}<\frac{\sqrt{A^2+8G^2}+A}{4}$*. This shows* $K=(x-y)\left(J/(LI)\right)_x<0$*.*

Moreover, $f(+0,1):=\lim_{x\to 0,x>0}f(x,1)=0$ *but* $\lim_{x\to\infty}xf(e^{-x},1)=1$*. According to part one of Theorem 8 the inequality*

$$\mathcal{H}_L(p,q;a,b)\le\mathcal{H}_L(r,s;a,b)$$

holds for all $(a,b)\in\mathbb{R}_+\times\mathbb{R}_+$ *if and only if both (1.2) and (1.3) hold. This is just the result proved first by Leach and Scholander in* [6]*.*

Example 2. *Let* $f(x,y)=A(x,y)=\frac{x+y}{2}(x,y>0,x\neq y)$*. Then* $\mathcal{H}_A(a,b;p,q)$ *is just the so-called Gini mean (1.5). By some partial derivative calculations, we get*

$$\begin{aligned}
I &= (\ln f)_{xy}=-\frac{1}{(x+y)^2}<0,\\
J &= (x-y)(xI_1)_x=\frac{(x-y)^2}{(x+y)^3},\\
J/(LI) &= -\frac{(x-y)\ln(x/y)}{x+y)}\overset{x/y=t}{=}-\frac{(t-1)\ln t}{t+1}\\
(J/(LI))_x &= -\frac{1}{y}\frac{\frac{t-1}{t}(t+1)+2\ln t}{(t+1)^2},\\
K &= (x-y)\left(J/(LI)\right)_x=-\frac{\frac{(t-1)^2}{t}(t+1)+2(t-1)\ln t}{(t+1)^2}<0.
\end{aligned}$$

Moreover, $f(+0,1):=\lim_{x\to 0,x>0}f(x,1)=\frac{1}{2}\neq 0$*. According to part two of Theorem 8 the inequality*

$$\mathcal{H}_A(p,q;a,b)\le\mathcal{H}_A(r,s;a,b)$$

holds for all $(a,b)\in\mathbb{R}_+\times\mathbb{R}_+$ *if and only if both (1.2) and (1.6) hold. This is just the comparison theorem for Gini means given by Páles's in* [7]*.*

Example 3. *Let* $f(x,y)=E(x,y)=e^{-1}\left(x^x/y^y\right)^{\frac{1}{x-y}}(x,y>0,x\neq y)$*. Then* $\mathcal{H}_E(a,b;p,q)$ *is defined by*

$$\mathcal{H}_E(p,q;a,b)=\begin{cases}\left(\dfrac{E(a^p,b^p)}{E(a^q,b^q)}\right)^{\frac{1}{p-q}} & p\neq q\\ G_{E,p}(a,b) & p=q\neq 0\\ G(a,b) & p=q=0\end{cases}. \tag{4.1}$$

By some partial derivative calculations, we get

$$\begin{aligned}
I &= (\ln f)_{xy}=\frac{1}{(x-y)^3}\left[2(x-y)-(x+y)\ln(x/y)\right]\\
&= \frac{2\ln(x/y)}{(x-y)^3}\left[L(x,y)-\frac{x+y}{2}\right].
\end{aligned}$$

From the well-known inequality $L(x,y) < \frac{x+y}{2}$ *we have* $I < 0$. *On the other hand,*

$$\begin{aligned}
J &= (x-y)(xI_1)_x = \frac{-3(x^2-y^2)+(x^2+4xy+y^2)(\ln x - \ln y)}{(x-y)^3},\\
J/(LI) &= \frac{\left[-3(x^2-y^2)+(x^2+4xy+y^2)\ln(x/y)\right]\ln(x/y)}{\left[2(x-y)-(x+y)\ln(x/y)\right](x-y)},\\
&\overset{x/y=t}{=} \frac{-3(t^2-1)\ln t + (t^2+4t+1^2)\ln^2 t}{2(t-1)^2 - (t^2-1)\ln t},\\
K &= (x-y)\left(J/(LI)\right)_x\\
&= (t-1)\frac{4t(t^2+t+1)\ln^3 t-(t^2-1)(t^2+16t+1)\ln^2 t+4(t-1)^2(t^2+7t+1)\ln t-6(t-1)^3(t+1)}{t\left[2(t-1)^2-(t^2-1)\ln t\right]^2}\\
&= \frac{(t-1)\ln^3 t}{t\left[2(t-1)^2-(t^2-1)\ln t\right]^2}\chi(L),
\end{aligned}$$

where $L = (t-1)/\ln t$ *and*

$$\chi(L) = 4t(t^2+t+1) - (t+1)(t^2+16t+1)L + 4(t^2+7t+1)L^2 - 6(t+1)L^3.$$

Set $A = (t+1)/2, G = \sqrt{t}$, *then* $\chi(L)$ *can be changed as*

$$\chi(L) = 4\left[G^2(4A^2-G^2) - A(2A^2+7G^2)L + (4A^2+5G^2)L^2 - 3AL^3\right].$$

Note

$$\begin{aligned}
\chi'(u) &= 4\left[-A(2A^2+7G^2) + 2(4A^2+5G^2)u - 9Au^2\right]\\
&= -36A\left(u - \frac{4A^2+5G^2}{9A}\right)^2 - \frac{4(A^2-G^2)(2A^2+25G^2)}{9A} < 0,
\end{aligned}$$

and

$$\begin{aligned}
\chi(A) &= 4(2A^2G^2 - A^4 - G^4) = -4(A^2-G^2)^2 \le 0,\\
\chi(A^{\frac{1}{3}}G^{\frac{2}{3}}) &= 4(2A^2G^2 - A^4 - G^4) = -G^{\frac{2}{3}}(A^{\frac{2}{3}} - G^{\frac{2}{3}})^3(2A^{\frac{4}{3}} + 2A^{\frac{2}{3}}G^{\frac{2}{3}} - G^{\frac{4}{3}}) \le 0,
\end{aligned}$$

by Lemma 1, we have $\chi(u) \le 0$ *for all* $u \in (A^{\frac{1}{3}}G^{\frac{2}{3}}, A)$. *It follows from* $A^{\frac{1}{3}}G^{\frac{2}{3}} < L < A$ $(t \ne 1)$*that* $\chi(L) < 0$, *and then*

$$K = \frac{(t-1)\ln^3 t}{t\left[2(t-1)^2-(t^2-1)\ln t\right]^2}\chi(L) < 0.$$

Moreover, $f(0,1) := \lim\limits_{x\to 0} f(x,1) = \exp\lim\limits_{x\to 0}\left(\frac{x\ln x}{x-1} - 1\right) = e^{-1} \ne 0$. *According to part two of Theorem 8 the inequality*

$$\mathcal{H}_E(p,q;a,b) \le \mathcal{H}_E(r,s;a,b)$$

holds for all $(a,b) \in \mathbb{R}_+ \times \mathbb{R}_+$ *if and only if both (1.2) and (1.6) hold.*

Example 4. *Let* $f(x,y) = D(x,y) = |x-y| (x,y>0, x\neq y)$. *Then* $\mathcal{H}_D(a,b;p,q)$ *is defined by*

$$\mathcal{H}_D(p,q;a,b) = \begin{cases} |\frac{a^p-b^p}{a^q-b^q}|^{\frac{1}{p-q}} & p\neq q, pq\neq 0 \\ G_{D,p}(a,b) & p=q\neq 0 \end{cases}, \tag{4.2}$$

where $G_{D,p}(a,b) = G_{D,p} = e^{\frac{1}{p}}E^{\frac{1}{p}}(a^p,b^p) = e^{\frac{1}{p}}E_p, E(a,b) = e^{-1}\left(a^a/b^b\right)^{1/(a-b)}$. *It is obvious that* $f(x,y)$ *is a homogeneous, symmetric and four times differentiable function satisfying the conditions that* $I<(>)0$ *and* $\operatorname{sgn} K$ *is invariant;* $(a,b),(p,q),(r,s)\in\mathbb{R}_+\times\mathbb{R}_+$, $f(1,1)=0$ *but* $f_x(+1,1) := \lim_{x\to 1, x>1} f_x(x,1) = 1$*; by simple calculations, we get*

$$\begin{aligned}
I &= (\ln f)_{xy} = \frac{1}{(x-y)^2} > 0\\
I(0,1) &: = \lim_{x\to 0} f(x,1) > 0\\
J &= (x-y)(xI_1)_x = -\frac{x+y}{(x-y)^2} < 0\\
J/(LI) &= -\frac{(x+y)\ln(x/y)}{x-y)} \overset{x/y=t}{=} -\frac{(t+1)\ln t}{t-1}\\
(J/(LI))_x &= -\frac{1}{y}\frac{(\frac{t+1}{t}+\ln t)(t-1)-(t+1)\ln t}{(t-1)^2}\\
&= -\frac{1}{y}\frac{\frac{t^2-1}{t}-\ln t^2}{(t-1)^2} = -\frac{\ln t^2}{ty(t-1)^2}\left(\frac{t^2-1}{\ln t^2}-\sqrt{t^2}\right)\\
K &= (x-y)\left(J/(LI)\right) = -\frac{2(t-1)\ln t}{ty(t-1)^2}\left(\frac{t^2-1}{\ln t^2}-\sqrt{t^2}\right) < 0,
\end{aligned}$$

in which $\frac{t^2-1}{\ln t^2} - \sqrt{t^2} > 0$ *follows from the well-known inequality* $L(x,y) > G(x,y)$.

Moreover, $f(+0,1) := \lim_{x\to 0, x>0} f(x,1) = 1 \neq 0$. *According to part two of Theorem 9 for* $p,q,r,s>0$ *the inequality*

$$\mathcal{H}_D(p,q;a,b) \leq \mathcal{H}_D(r,s;a,b)$$

holds for all $(a,b)\in\mathbb{R}_+\times\mathbb{R}_+$ *if and only if the following both inequalities*

$$\begin{cases} (i) & l(p,q)\geq l(r,s),\\ (ii) & \min(p,q)\geq \min(r,s)\end{cases} \tag{4.3}$$

hold.

By comparison theorems for extended mean and Gini mean, for $p,q,r,s>0$ all the following inequalities hold

$$\mathcal{H}_L(p,q;a,b) \geq \mathcal{H}_L(r,s;a,b), \tag{4.4}$$

$$\mathcal{H}_A(p,q;a,b) \geq \mathcal{H}_A(r,s;a,b), \tag{4.5}$$

$$\mathcal{H}_D(p,q;a,b) \leq \mathcal{H}_D(r,s;a,b) \tag{4.6}$$

if and only if $p+q \geq r+s$ and (4.3) hold. Note

$$\mathcal{H}_D(p,q;a,b) = (p/q)^{1/(p-q)}\mathcal{H}_L(p,q;a,b)$$

from (4.4) and (4.6) it follows that

$$1 \leq \frac{\mathcal{H}_L(p,q;a,b)}{\mathcal{H}_L(r,s;a,b)} \leq \frac{(r/s)^{1/(r-s)}}{(p/q)^{1/(p-q)}}. \tag{4.7}$$

Note $\mathcal{H}_D^2(2p,2q;a,b) = \mathcal{H}_D(p,q;a,b)\mathcal{H}_A(p,q;a,b)$, from $\mathcal{H}_D^2(2p,2q;a,b) \leq \mathcal{H}_D^2(2p,2q;a,b)$ it follows further that

$$1 \leq \frac{\mathcal{H}_L(p,q;a,b)}{\mathcal{H}_L(r,s;a,b)} \leq \frac{(r/s)^{1/(r-s)}}{(p/q)^{1/(p-q)}}\frac{\mathcal{H}_A(r,s;a,b)}{\mathcal{H}_A(p,q;a,b)} \leq \frac{(r/s)^{1/(r-s)}}{(p/q)^{1/(p-q)}}. \tag{4.8}$$

The results above can be stated as follows:

Corollary 1. [1, Corollary 1.2]*Let $p,q,r,s > 0$. Then inequalities (4.8) hold if and only if all the $p+q \geq r+s$ and (4.3) hold.*

Remark 1. *In* [1], *a condition that $l(p,q) \geq l(r,s)$ of Corollary 1.2 is left out. It is absolutely necessary.*

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